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THE DESIGN AND IMPLEMENTATION OF A COMPUTATIONAL ALGORITHM OF GOMPERTZ - LOGISTIC MODELS ON ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT: *This Paper presents the design and Implementation of a Numerical Model for solving ordinary differential equations with Initial values. It was a resultant equation of Gompertz and Logistic models. Some problems were solved successfully to test the accuracy, performance and suitability of the Model in terms of the absolute relative errors computed via MATLAB codes. The numerical solutions obtained via the Model agreed with the analytical solutions of the problems that were examined. The analysis of the properties of the method such as consistency, stability and convergence were also investigated. It was found to converge, be consistent and very stable. The combinations of Gompertz and Logistic Models as the basis function makes the work an improvement of existing methods used to evaluate growth problems and IVPs of various areas of Ordinary Differential Equations (ODEs).*

Keywords: Numerical model, Basis Functions, Computational Algorithm, Gompertz equation, and Logistic equation.

1.0 Introduction:

The Mathematical Formulation of problems that occurs in divers' real and physical phenomena like determining the charge or current in an electric circuit; the study of the Rate of decomposition of radioactive substance; the rate of growth of a population, the study of the reactions of chemical kinetics among others; give rise to differential equations and each object involved must obey certain scientific laws. These laws involved various rates of change of one or more quantities with respect to other quantities and these rates of change are expressed mathematically by derivatives. Most of these problems require the solution of an initial-value problem, that is, the solution to a differential equation that satisfies a given initial conditions. In common real-life situations, the differential equation that models the problem is too complicated to solve explicitly, and approaches are taken to approximate the solution. The approximation method gives more accurate results and maintains a reasonable error-bound. The numerical solution of first order initial value problems have caught much attention recently where most of these integrators (algorithms/methods) were developed by representing the actual solution to initial value problem by an interpolating function. This paper considers the numerical solution to initial value problems of the form.

$$y' = f(x, y); \qquad y(x_0) = y_0 \qquad (1)$$

The approach was based on a local representation of the solution by a combination of a polynomial, oscillatory, and exponential basis functions as interpolant.

2.0 Development of the Numerical Model

A Numerical tool of the form,

$$y_{n+1} = y_n + h\emptyset(x_n, y_n; h) \quad (2)$$

as a solution to the initial value problems of ordinary differential equations is developed from the combination of the second type of Gompertz equation of the form

$$f(x) = ae^{be^{cx}} + dg^x \quad (3)$$

Where:

a is an asymptote, since $f(x) = ae^{be^{\infty}}$ $=ae^0= a$, as x approaches ∞

b sets the displacement along the x axis (translates the graph to the left or right)

c sets the growth rate (y scaling), b and c are positive numbers, d and g are constants.

e is constant (e = 2.71828….).

and Logistic function or Logistic curve (sometimes called the Verhulst Model or Logistic growth curve) which is a model for population growth rate first published by Pierre Verhulst (1845,1847) given as:

$$f(x) = \frac{k}{1+ae^{bx}} \quad (4)$$

Where a, b, and K are constants, K is called the carrying capacity.

It is known that if $a > 0$, the logistic model decreases and increases when b > 0

The basis function below was developed from the combination of (3) and (4) with trigonometric function,

Multiplication of e^x and $\frac{1}{1+e^x} = \frac{e^x}{e^x+1}$, divide through by $e^x = \frac{1}{1+e^{-x}}$

$$F(x) = \frac{\alpha}{1+e^{-x}} + \beta C^x + \gamma cosx \quad (5)$$

where $x \in [0,1]$, $\alpha, \beta, \gamma,$ are undetermined coefficients, $and\ C$ is the determined constant known as Shape parameter according to Gompertz and Logistic models.

Consider the theoretical solution $y(x)$ to the initial value problem of ordinary differential equation (1) in the interval $[x_n, x_{n+1}]$. Therefore, y_n is the numerical estimate to the theoretical solution $y(x)$.

The mesh points are defined as

$$x_n = a + nh,$$
$$x_{n+1} = a + (n+1)h,\ n = 0,1,2, \dots \dots \quad (6)$$

The undetermined coefficients are valued by imposing some constraints on the polynomial as follows:

(a) The basis function must coincide with the theoretical solution at $x = x_n$ and the increment $x = x_{n+1}$, which required

$$F(x_n) = \frac{\alpha}{1+e^{-x_n}} + \beta c^{x_n} + \gamma cosx_n \quad (7)$$

and $$F(x_{n+1}) = \frac{\alpha}{1+e^{-x_{n+1}}} + \beta c^{x_{n+1}} + \gamma cosx_{n+1} \quad (8)$$

(b) Obtained the derivatives of the basis function that coincide with the differential equation as well as its first, second, and third derivatives with respect to x at $x = x_n$.

Denote the ith total derivatives of $f(x)$ with respect to x with $F's$ Such that

$$F(x_n) = \frac{\alpha}{1+e^{-x_n}} + \beta c^{x_n} + \gamma cos x_n$$

$$F'(x_n) = FN = f_n = \frac{\alpha e^{-x_n}}{(1+e^{-x_n})^2} + \beta c^{x_n} \ln(\mathrm{C}) - \gamma \sin x_n$$

$$F''(x_n) = FN1 = f_n' = \frac{2\alpha(e^{-x_n})^2}{(1+e^{-x_n})^3} - \frac{\alpha e^{-x_n}}{(1+e^{-x_n})^2} + \beta c^{x_n} \ln(\mathrm{C})^2 - \gamma \cos x_n$$

$$F'''(x_n) = FN2 = f_n^2 = \frac{6\alpha(e^{-x_n})^3}{(1+e^{-x_n})^4} - \frac{6\alpha(e^{-x_n})^2}{(1+e^{-x_n})^3} + \frac{\alpha e^{-x_n}}{(1+e^{-x_n})^2} + \beta c^{x_n} \ln(\mathrm{C})^3 + \gamma \sin x_n \qquad (9)$$

Taking this into linear equation using the Crammers rule method, we have

$$\begin{bmatrix} \frac{e^{-x_n}}{(1+e^{-x_n})^2} & C^{x_n} \ln(\mathrm{C}) & -\sin x_n \\ \frac{2(e^{-x_n})^2}{(1+e^{-x_n})^3} - \frac{e^{-x_n}}{(1+e^{-x_n})^2} & C^{x_n} \ln(\mathrm{C})^2 & -\cos x_n \\ \frac{6(e^{-x_n})^3}{(1+e^{-x_n})^4} - \frac{6(e^{-x_n})^2}{(1+e^{-x_n})^3} + \frac{e^{-x_n}}{(1+e^{-x_n})^2} & C^{x_n} \ln(\mathrm{C})^3 & \sin x_n \end{bmatrix} X \quad \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} f_n \\ f_n^1 \\ f_n^2 \end{bmatrix}$$

Then

$$\begin{aligned} \alpha = \ & -((FNC^{x_n} \ln(\mathrm{C})^3 \cos(x_n) - FN1C^{x_n} \ln(\mathrm{C})^3 \sin(x_n) + FNC^{x_n} \ln(\mathrm{C})^2 \sin(x_n) \\ & +FN2C^{x_n} \sin(x_n) \ln(\mathrm{C})^2 - FN1C^{x_n} \ln(\mathrm{C}) \sin(x_n) - FN2C^{x_n} \ln(\mathrm{C}) \cos(x_n)(1+e^{-x_n})^4 / \\ & (e^{-x_n} C^{x_n} \ln(\mathrm{C}) (\ln(\mathrm{C})^2 \sin(x_n) (\mathrm{e}^{-x_n})^2 - \ln(\mathrm{C})^2 \cos(x_n)(\mathrm{e}^{-x_n})^2 - 2\ln(\mathrm{C}) \sin(x_n) (\mathrm{e}^{-x_n})^2 \\ & -2\ln(\mathrm{C})^2 \cos(x_n)\, e^{-x_n} - \ln(\mathrm{C})^2 \sin(x_n) + 2\ln(\mathrm{C}) \sin(x_n) e^{-x_n} + \sin(x_n) (\mathrm{e}^{-x_n})^2 \\ & -\ln(\mathrm{C})^2 \cos(x_n) + \cos(x_n)(\mathrm{e}^{-x_n})^2 - 2\ln(\mathrm{C}) \sin(x_n) - 4\cos(x_n) e^{-x_n} - \sin(x_n) + cos(x_n))) \end{aligned}$$

$$\begin{aligned} \beta = \ & (FN\sin(x_n)(e^{-x_n})^2 + FN\cos(x_n)(e^{-x_n})^2 - 2FN1\sin(x_n)(e^{-x_n})^2 \\ & +FN2\sin(x_n)(e^{-x_n})^2 - FN2\cos(x_n)(e^{-x_n})^2 - 4FN\cos(x_n)e^{-x_n} \\ & +2FN1\sin(x_n)e^{-x_n} - 2FN2\cos(x_n)e^{-x_n} - FN\sin(x_n) \\ & +FNcos(x_n) - 2FN1\sin(x_n) - FN2\sin(x_n) - FN2cos(x_n)) \\ & /(C^{x_n} \ln(\mathrm{C}) ((\ln(\mathrm{C})^2 \sin(x_n) (\mathrm{e}^{-x_n})^2 - \ln(\mathrm{C})^2 \cos(x_n)(\mathrm{e}^{-x_n})^2 - 2\ln(\mathrm{C}) \sin(x_n) (\mathrm{e}^{-x_n})^2 - \\ & 2\ln(\mathrm{C})^2 \cos(x_n)\, e^{-x_n} - \ln(\mathrm{C})^2 \sin(x_n) + 2\ln(\mathrm{C}) \sin(x_n) e^{-x_n} + \sin(x_n) (\mathrm{e}^{-x_n})^2 \\ & -\ln(\mathrm{C})^2 \cos(x_n) + \cos(x_n)(\mathrm{e}^{-x_n})^2 - 2\ln(\mathrm{C}) \sin(x_n) - 4\cos(x_n) e^{-x_n} - \sin(x_n) + cos(x_n))) \end{aligned}$$

$$\begin{aligned} \gamma = \ & -(FN \ln(\mathrm{C})^2 (e^{-x_n})^2 - FN1(e^{-x_n})^2 \ln(\mathrm{C})^2 - \mathrm{FN}(e^{-x_n})^2 \ln(\mathrm{C}) - 2\mathrm{FN1}e^{-x_n} \ln(\mathrm{C})^2 + \mathrm{FN2}(e^{-x_n})^2 \ln(\mathrm{C}) - \\ & \ln(\mathrm{C})^2 FN + 4FNe^{-x_n} \ln(\mathrm{C}) - \ln(\mathrm{C})^2 FN1 + \mathrm{FN1}(e^{-x_n})^2 \\ & +2FN2e^{-x_n} \ln(\mathrm{C}) - \mathrm{FN2}(e^{-x_n})^2 - \ln(\mathrm{C})\, \mathrm{FN} - 4FN1e^{-x_n} + \ln(\mathrm{C})\, \mathrm{FN2} + \mathrm{FN1} + \mathrm{FN2}) / \\ & (\sin(x_n)(e^{-x_n})^2 \ln(\mathrm{C})^2 - \cos(x_n)(e^{-x_n})^2 \ln(\mathrm{C})^2 - 2\sin(x_n) \ln(\mathrm{C}) (e^{-x_n})^2 - \\ & 2\cos(x_n)e^{-x_n} \ln(\mathrm{C})^2 - \sin(x_n) \ln(\mathrm{C})^2 + 2\sin(x_n) \ln(\mathrm{C}) e^{-x_n} + \sin(x_n)(e^{-x_n})^2 - cos(x_n) \ln(\mathrm{C})^2 + \\ & \cos(x_n)(e^{-x_n})^2 - 2\sin(x_n) \ln(\mathrm{C}) - 4cos(x_n)e^{-x_n} - \sin(x_n) + \cos(x_n) \qquad (10) \end{aligned}$$

Therefore, in general

$$F(x_{n+1}) - F(x_n) = y_{n+1} - y_n$$

$$= \alpha\left(\frac{1}{1+e^{-x_{n+1}}} - \frac{1}{1+e^{-x_n}}\right) + \beta\left(c^{x_{n+1}} - c^{x_n}\right) + \gamma\left(cosx_{n+1} - cosx_n\right) \tag{11}$$

By simplification, $\frac{1}{1+e^{-x_{n+1}}} - \frac{1}{1+e^{-x_n}} = \frac{1-e^{-h}}{1+e^{x_n}+e^{-h}+e^{-x_n-h}}$

$$c^{x_{n+1}} - c^{x_n} = c^{x_n}[c^h - 1]$$

$$cosx_{n+1} - cosx_n = cosx_n(cosh-1) - sinx_n sinh \tag{12}$$

Hence,

$$y_{n+1} = y_n + \alpha\left(\frac{1-e^{-h}}{1+e^{x_n}+e^{-h}+e^{-x_n-h}}\right) + \beta\left(c^{x_n}[c^h-1]\right) + \gamma\left(cosx_n(cosh-1) - sinx_n sinh\right). \tag{13}$$

Which gives the new numerical method for the solution of first order differential equation.

$$
\begin{aligned}
y_{n+1} = y_n &+ \frac{\begin{array}{c} sinx_n\left[(logc)^2 f_n - logc f_n{}^1 - (logc)^3 f_n{}^1 + (logc)^2 f_n{}^2\right] \\ +cosx_n\left[(logc)^3 f_n{}^1 - logc f_n{}^1\right]\left\{\frac{1-e^{-h}}{1+e^{x_n}+e^{-h}+e^{-x_n-h}}\right\}\end{array}}{\begin{array}{c} sinx_n\left[\begin{array}{c}\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^2 - \frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}logc + \frac{e^{-x_n}}{(e^{-x_n}+1)^2}logc - \frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}(logc)^3 \\ + \frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^3 + \frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^2 - \frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}(logc)^2 + \frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}(logc)^2\end{array}\right] \\ +cosx_n\left[\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^3 - \frac{e^{-x_n}}{(e^{-x_n}+1)^2}logc + \frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}logc - \frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}logc\right]\end{array}} \\
&+ \frac{\begin{array}{c} sinx_n\left[\begin{array}{c}\frac{e^{-x_n}}{(e^{-x_n}+1)^2}f_n{}^1 - \frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}f_n + \frac{e^{-x_n}}{(e^{-x_n}+1)^2}f_n - \frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}f_n{}^2 \\ + \frac{e^{-x_n}}{(e^{-x_n}+1)^2}f_n{}^2 + \frac{e^{-x_n}}{(e^{-x_n}+1)^2}f_n{}^1 - \frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}f_n{}^1 + \frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}f_n{}^1\end{array}\right] \\ +cosx_n\left[\frac{e^{-x_n}}{(e^{-x_n}+1)^2}f_n{}^2 - \frac{e^{-x_n}}{(e^{-x_n}+1)^2}f_n + \frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}f_n - \frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}f_n\right]\left\{c^h-1\right\}\end{array}}{\begin{array}{c} sinx_n\left[\begin{array}{c}\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^2 - \frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}logc + \frac{e^{-x_n}}{(e^{-x_n}+1)^2}logc - \frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}(logc)^3 \\ + \frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^3 + \frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^2 - \frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}(logc)^2 + \frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}(logc)^2\end{array}\right] \\ +cosx_n\left[\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^3 - \frac{e^{-x_n}}{(e^{-x_n}+1)^2}logc + \frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}logc - \frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}logc\right]\end{array}} \\
&+ \frac{\begin{array}{c} f_n\left[\begin{array}{c}\frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}(logc)^3 - \frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^3 - \frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^2 \\ + \frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}(logc)^2 - \frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}(logc)^2\end{array}\right] \\ -f_n{}^1\left[\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^3 - \frac{e^{-x_n}}{(e^{-x_n}+1)^2}logc + \frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}logc - \frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}logc\right] \\ +f_n{}^2\left[\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^2 - \frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}logc + \frac{e^{-x_n}}{(e^{-x_n}+1)^2}logc\right] \\ \left\{cosx_n(cosh-1) - sinx_n sinh\right\}\end{array}}{\begin{array}{c} sinx_n\left[\begin{array}{c}\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^2 - \frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}logc + \frac{e^{-x_n}}{(e^{-x_n}+1)^2}logc - \frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}(logc)^3 \\ + \frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^3 + \frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^2 - \frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}(logc)^2 + \frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}(logc)^2\end{array}\right] \\ +cosx_n\left[\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^3 - \frac{e^{-x_n}}{(e^{-x_n}+1)^2}logc + \frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}logc - \frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}logc\right]\end{array}}
\end{aligned} \tag{14}
$$

equation (14) gives the derived Numerical Method.

3.0 Analysis of the Properties of the Numerical Method

Definition 1 (Henrici ,1962)

Any algorithm for solving ordinary differential equations in which the approximation y_{n+1} to the solution at the point x_{n+1} can be calculated if x_n, y_n and h are known as one-step method. It is a common practice to write the function dependence y_{n+1}, as the quantities of x_n, y_n and h in the form:

$$y_{n+1} = y_n + h\emptyset(x_n, y_n; h) \qquad (15)$$

Therefore, the increment function

$$\emptyset(x_n, y_n; h) = \{Gf_n + Kf_n^1 + Lf_n^2\} \qquad (16)$$

Theorem 1 (Henrici,1962); Lipschitz condition

Let the increment function of the method (14) defined by (16) be continuous as a function of its arguments in the region defined by:

$$x \in [a, b], y \in (-\infty, \infty); \;\; 0 \le h \le h_o, \qquad (17)$$

where, $h_o > 0$, let there exists a constant L such that

$|\emptyset(x_n, y_n^*; h) - \emptyset(x_n, y_n; h)| \le L|y_n^* - y_n|$ for all $(x_n, y_n; h)$and $(x_n, y_n^*; h)$

in the region just defined is a necessary and sufficient condition for the convergence of the method.

3.1 Convergency of the numerical method

Consider the derived numerical method (14) The theoretical solution $y(x)$ to the initial value problem of ordinary differential equation of the form (1) in the interval $[x_n, x_{n+1}]$.

Expanding the following to the fifth terms in Taylor's series:

$$M = \frac{1 - e^{-h}}{1 + e^{x_n} + e^{-h} + e^{-x_n - h}}$$

Put $x_n = nh$

$$M = \frac{1 - e^{-h}}{1 + e^{nh} + e^{-h} + e^{-nh-h}} = \frac{1 - e^{-h}}{1 + e^{nh} + e^{-h} + e^{-(nh+h)}}$$

$$= \frac{-h + \frac{h^2}{2!} - \frac{h^3}{3!} + \frac{h^4}{4!} - \frac{h^5}{5!}}{1 + 1 + nh + \frac{(nh)^2}{2!} + \frac{(nh)^3}{3!} + \frac{(nh)^4}{4!} + 1 - h + \frac{h^2}{2!} - \frac{h^3}{3!} + \frac{h^4}{4!} + 1 - (nh + h) + \frac{(nh+h)^2}{2!} - \frac{(nh+h)^3}{3!} + \frac{(nh+h)^4}{4!}}$$

$$= \frac{-h + \frac{h^2}{2!} - \frac{h^3}{3!} + \frac{h^4}{4!} - \frac{h^5}{5!}}{4 - 2h + h^2 + nh^2 + (nh)^2 - \frac{h^3}{3} - \frac{nh^3}{2} - \frac{(nh)^2}{2} + \frac{h^4}{12} + \frac{nh^4}{6} + \frac{(nh)^4}{24} + \frac{n^2h^4}{4} + \frac{n^3h^4}{6}} \qquad (18)$$

$$N = c^h - 1 \Longrightarrow \sum_{k=0}^{\infty} h^k (logc)^k - 1 = h\, logc + h^2 (logc)^2 + h^3 (logc)^3 + h^4 (logc)^4 \qquad (19)$$

And $S = Cosx_n(cosh - 1) - sinx_n\, sinh$

$$= cosx_n\left(\sum_{k=0}^{\infty}\frac{h^{2k}}{(2k)!}-1\right)-sinx_n\left(\sum_{k=0}^{\infty}\frac{h^{(2k+1)}}{(2k+1)!}\right)$$

$$=\frac{h^2}{2!}cosx_n+\frac{h^4}{4!}cosx_n+\frac{h^6}{6!}cosx_n+\frac{h^8}{8!}cosx_n-\left(h\, sinx_n+\frac{h^3}{3!}sinx_n+\frac{h^5}{5!}sinx_n+\frac{h^7}{7!}sinx_n\right) \quad (20)$$

Put [18,19 and 20 into 14], and let

$$P=\frac{\begin{array}{c} sinx_n\left[(logc)^2 f_n-logc f_n{}^1-(logc)^3 f_n{}^1+(logc)^2 f_n{}^2\right] \\ +cosx_n\left[(logc)^3 f_n{}^1-logc f_n{}^1\right]\end{array}}{\begin{array}{c} sinx_n\left[\begin{array}{c}\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^2-\frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}logc+\frac{e^{-x_n}}{(e^{-x_n}+1)^2}logc-\frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}(logc)^3 \\ +\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^3+\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^2-\frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}(logc)^2+\frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}(logc)^2\end{array}\right] \\ +cosx_n\left[\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^3-\frac{e^{-x_n}}{(e^{-x_n}+1)^2}logc+\frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}logc-\frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}logc\right]\end{array}}$$

$$Q=\frac{\begin{array}{c} sinx_n\left[\begin{array}{c}\frac{e^{-x_n}}{(e^{-x_n}+1)^2}f_n{}^1-\frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}f_n+\frac{e^{-x_n}}{(e^{-x_n}+1)^2}f_n-\frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}f_n{}^2 \\ +\frac{e^{-x_n}}{(e^{-x_n}+1)^2}f_n{}^2+\frac{e^{-x_n}}{(e^{-x_n}+1)^2}f_n{}^1-\frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}f_n{}^1+\frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}f_n{}^1\end{array}\right] \\ +cosx_n\left[\frac{e^{-x_n}}{(e^{-x_n}+1)^2}f_n{}^2-\frac{e^{-x_n}}{(e^{-x_n}+1)^2}f_n+\frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}f_n-\frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}f_n\right]\end{array}}{\begin{array}{c} sinx_n\left[\begin{array}{c}\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^2-\frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}logc+\frac{e^{-x_n}}{(e^{-x_n}+1)^2}logc-\frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}(logc)^3 \\ +\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^3+\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^2-\frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}(logc)^2+\frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}(logc)^2\end{array}\right] \\ +cosx_n\left[\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^3-\frac{e^{-x_n}}{(e^{-x_n}+1)^2}logc+\frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}logc-\frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}logc\right]\end{array}}$$

$$R=\frac{\begin{array}{c} f_n\left[\begin{array}{c}\frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}(logc)^3-\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^3-\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^2 \\ +\frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}(logc)^2-\frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}(logc)^2\end{array}\right] \\ -f_n{}^1\left[\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^3-\frac{e^{-x_n}}{(e^{-x_n}+1)^2}logc+\frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}logc-\frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}logc\right] \\ +f_n{}^2\left[\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^2-\frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}logc+\frac{e^{-x_n}}{(e^{-x_n}+1)^2}logc\right]\end{array}}{\begin{array}{c} sinx_n\left[\begin{array}{c}\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^2-\frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}logc+\frac{e^{-x_n}}{(e^{-x_n}+1)^2}logc-\frac{2e^{-2x_n}}{(e^{-x_n}+1)^3}(logc)^3 \\ +\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^3+\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^2-\frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}(logc)^2+\frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}(logc)^2\end{array}\right] \\ +cosx_n\left[\frac{e^{-x_n}}{(e^{-x_n}+1)^2}(logc)^3-\frac{e^{-x_n}}{(e^{-x_n}+1)^2}logc+\frac{6e^{-2x_n}}{(e^{-x_n}+1)^3}logc-\frac{6e^{-3x_n}}{(e^{-x_n}+1)^4}logc\right]\end{array}}$$

(21)

Therefore,

$A = PM, D = QN \ and \ E = RS$

Thus, the numerical method (14) can be written compactly as

$$y_{n+1} = y_n + h\{Af_n + Df_n^1 + Ef_n^2\} \tag{22}$$

The above expression (22) converges since we can write it in the form

$$y_{n+1} = y_n + h\,\emptyset(x_n, y_n; h)$$
$$where \ \ \emptyset(x_n, y_n; h) = \{Af_n + Df_n^1 + Ef_n^2\} \tag{23}$$

which is known as the increment function.

3.2 Consistency of the Numerical Method

Definition 2 (Fatunla,1988)

A numerical scheme with an increment function $\emptyset(x_n, y_n; h)$ is said to be consistent with the initial value problem,

$y' = f(x, y), \ \ y(x_o) = y_o$ if $\emptyset(x_n, y_n; h) = f(x_n, y_n),$ (24)

and the method has order of at least one. Therefore, the numerical method (14) is consistent since it can be written as equation (24), Then the relation:

$$\emptyset(x_n, y_n; h) = f(x_n, y_n) \text{ as } h \ tends \ to \ 0$$

Hence,

$$\emptyset(x_n, y_n; h) = f(x_n, y_n)$$

3.3 Stability Analysis of the Numerical Method

Following the theorem stated by Fatunla (1988),

Let $y_n = y(x_n)$ and $P_n = P(x_n)$ (25)

denotes two different numerical solutions of initial value problems of ordinary differential equation

$$y' = f(x, y), \ \ y(x_o) = y_o$$

with the initial conditions specified as

$y(x_o) = \eta$ and $p(x_o) = \eta^*$ respectively, such that $|\eta - \eta^*| < \varepsilon, \ \ \varepsilon > 0.$

the Numerical method (14), and the increment function (22) was obtained

and written compactly as

$$y_{n+1} = y_n + h\{Pf_n + Qf_n^1 + Rf_n^2\} \tag{26}$$

Then, by expansion it gives

$$y_{n+1} = y_n + f_n h + \tfrac{1}{2} f_n^1 h^2 + \tfrac{1}{6} f_n^2 h^3 \tag{27}$$

Using the test equation

$y' = \lambda y, \qquad y(x_0) = y_0$ where $\lambda = u + iv$ (28)

$y_n' = \lambda y_n = f_n, \quad y_n'' = \lambda^2 y_n = f_n^1, \ \ y_n''' = \lambda^3 y_n = f_n^2$ e.t.c. (29)

Substitute (29) into (27) and expand gives

$$y_{n+1} = y_n(1 + \lambda h + \tfrac{1}{2}\lambda^2 h^2 + \tfrac{1}{6}\lambda^3 h^3) \tag{30}$$

To obtain

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$$\frac{y_{n+1}}{y_n} = 1 + \lambda h + \frac{1}{2}\lambda^2 h^2 + \frac{1}{6}\lambda^3 h^3 \tag{31}$$

Let $z = \lambda h$, and $\mu(z) = \frac{y_{n+1}}{y_n}$ therefore

$$\mu(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 \tag{32}$$

is the stability function of the method (14).

Definition 3 (Fatunla,1988)

The one - step method (14) is said to be absolutely stable at a point z in the complex plane provided the stability function or polynomial $\mu(z)$ of (32) fulfills the conditions $|\mu(z)| < 1$ and the region of absolute stability is defined as

$$RAS = \{z: |\mu(z)| < 1\} \tag{33}$$

Therefore the Region of Absolute Stability for the Numerical Method (14) is verified at $-2.5 \leq u \leq 0.5$ and $-2.5 < v < 2.5$, in the region $0 \leq \theta < 2 * pi$, Which confirms the A-stability in view of the definition and it can be seen also from the graph shown below

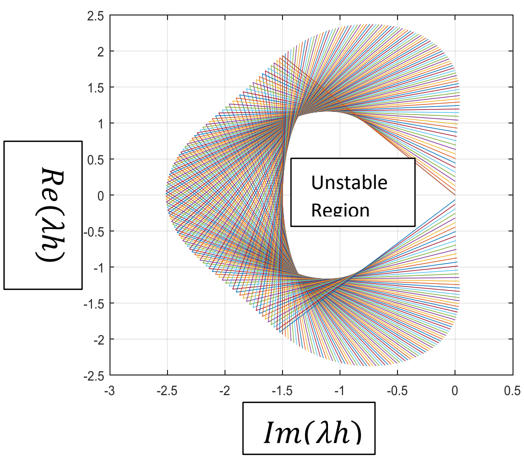

Figure 1: Region of Absolute Stability (shaded) for the Numerical Method.

4.0 IMPLEMENTATION AND RESULT

Problem 1: Solving for the Initial Value Problems

$y' = y; y(0) = 1$; the analytical solution $y(x) = exp(x)$,

$0 \leq x \leq 1, c = 0.008, h = 0.1$

Using the method (14) to solve the problem 1 and the results presented below.

Table 1

Xn	Numerical Solution	Analytical Solution	Absolute Error
[0.00]	[1.000000000000000]	[1.000000000000000]	[0.0000000000000000000]
[0.10]	[1.105153899372500]	[1.105170918075648]	[0.0000017018703147408]
[0.20]	[1.221368386484937]	[1.221402758160170]	[0.0000034371675233347]
[0.30]	[1.349806659091007]	[1.349858807576003]	[0.0000052148484996239]
[0.40]	[1.491754195912905]	[1.491824697641270]	[0.0000070501728365536]
[0.50]	[1.648631623657364]	[1.648721270700128]	[0.0000089647042763818]
[0.60]	[1.822008936380641]	[1.822118800390509]	[0.0000109864009868455]
[0.70]	[2.013621208180241]	[2.013752707470477]	[0.0000131499290235935]
[0.80]	[2.225385955408366]	[2.225540928492468]	[0.0000154973084101506]
[0.90]	[2.459422321189574]	[2.459603111156950]	[0.0000180789967375805]
[1.00]	[2.718072273079310]	[2.718281828459046]	[0.0000209555379735438]

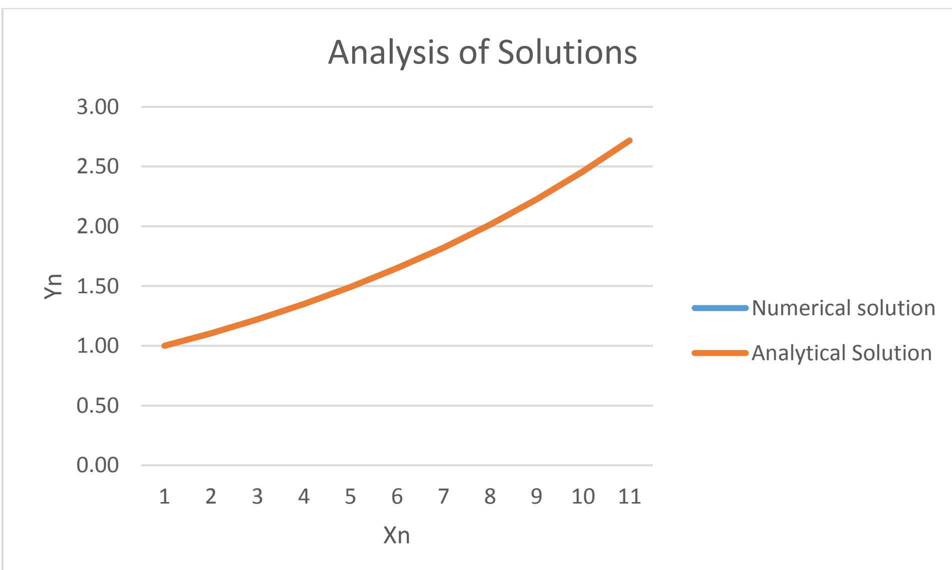

Figure 2: Analysis of Numerical and Analytical Solutions of $y' = y$ on line graph

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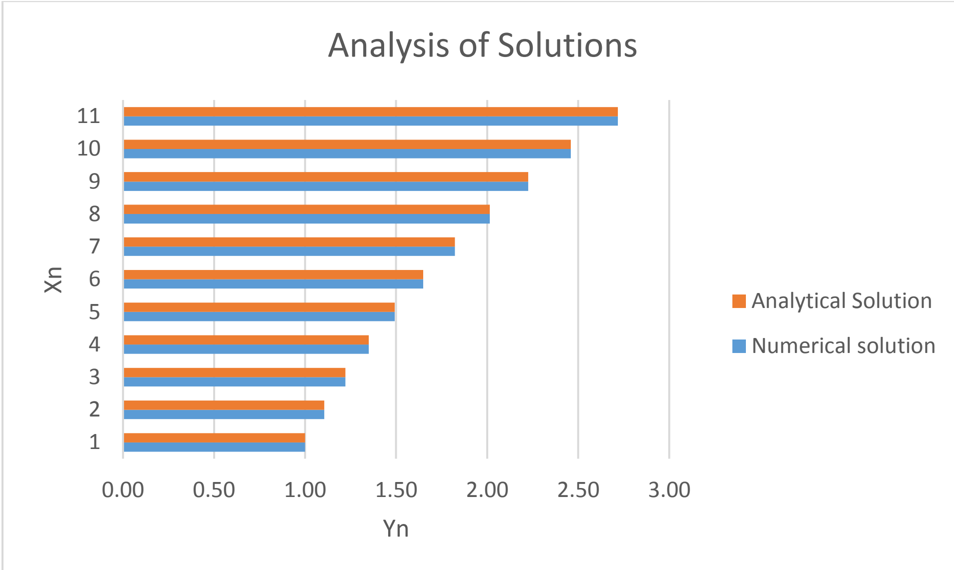

Figure 3: Analysis of Numerical and Analytical Solutions $y' = y$ on bar graph

<u>Problem 2</u>.

Solving for the Initial Value Problems

$y' = secx; y(0) = 1$; the analytical solution $y(x) = \ln(\sec(x) + \tan(x)$,

$0 \leq x \leq 1 \,, c = 0.01, h = 0.1$

Using the method (14) to solve the problem 2 and the results presented below.

Table 2

Xn	Numerical Solution	Analytical Solution	Absolute Error
[0.00]	[1.000000000000000]	[1.000000000000000]	[0.00000000000000]
[0.10]	[1.100273909013314]	[1.100167084547480]	[0.000106824465834]
[0.20]	[1.201725548466430]	[1.201346823567728]	[0.000378724898703]
[0.30]	[1.305396363892750]	[1.304603974401704]	[0.000792389491046]

[0.40]	[1.412451181902965]	[1.411114219868597]	[0.001336962034368]
[0.50]	[1.524248844022249]	[1.522238103278440]	[0.002010740743809]
[0.60]	[1.642442289209071]	[1.639622515807765]	[0.002819773401306]
[0.70]	[1.769128094101802]	[1.765350458597683]	[0.003777635504118]
[0.80]	[1.907082703865177]	[1.902176698545870]	[0.004906005319307]
[0.90]	[2.060158629116103]	[2.053923113741086]	[0.006235515375018]
[1.00]	[2.233996269866353]	[2.226191170883517]	[0.007805098982836]

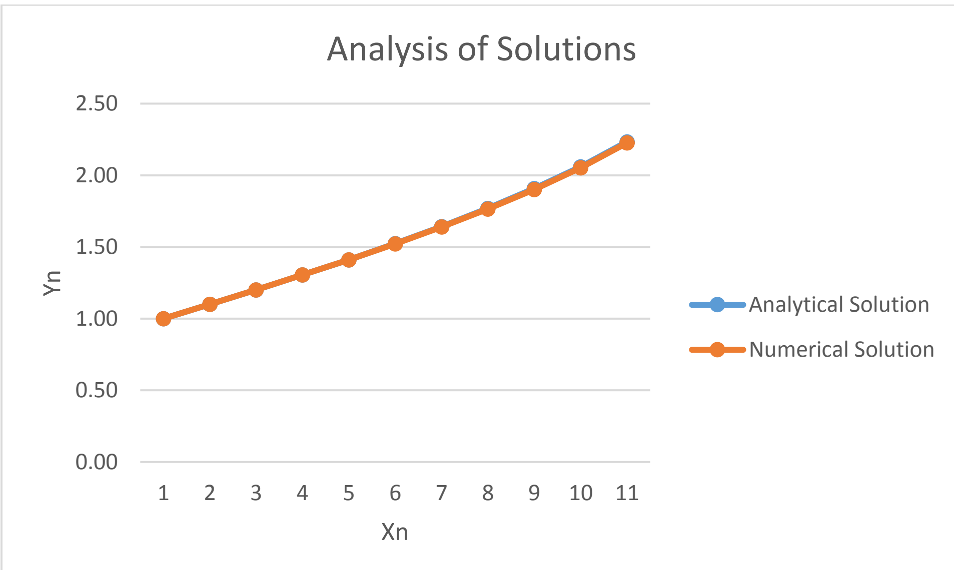

Figure 4: Analysis of Numerical and Analytical Solutions of $y' = sec(x)$ on line graph

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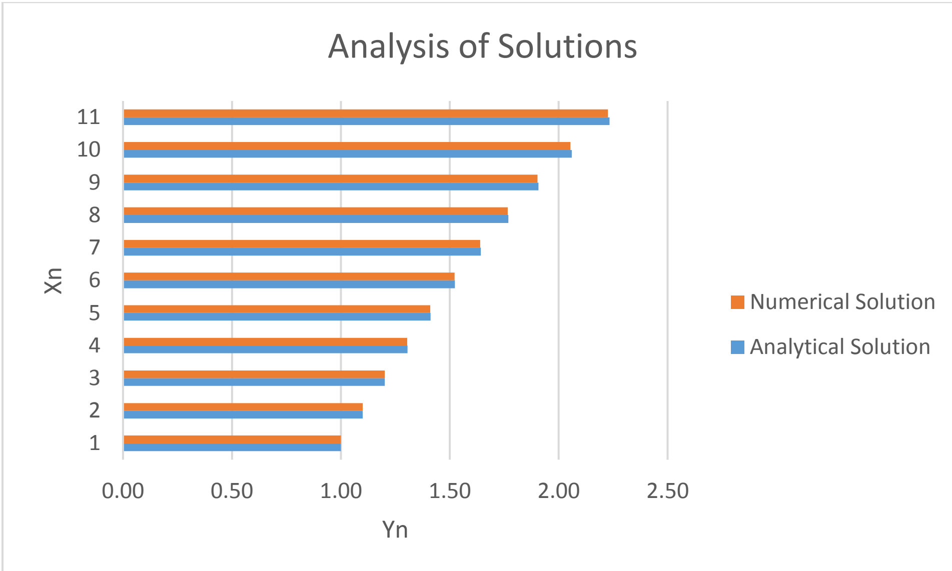

Figure 5: Analysis of Numerical and Analytical Solutions of $y' = sexc(x)$on bar graph

Problem 3.

Solving for the Initial Value Problems

$y' = x + y; y(0) = 1$; the analytical solution $y(x) = 2\exp(x - 1) - x - 1.$

$0 \leq x \leq 1, c = 0.0006, h = 0.1$

Using the method (14) to solve the problem 3 and the results presented below.

Table 3

Xn	Numerical Solution	Analytical Solution	Absolute Error
[0.00]	[1.000000000000000]	[1.000000000000000]	[0.0000000000000000]
[0.10]	[1.110348019113990]	[1.110341836151295]	[0.0000006182962694]
[0.20]	[1.242831064995378]	[1.242805516320340]	[0.0000025548675038]
[0.30]	[1.399779517489962]	[1.399717615152007]	[0.0000060336596954]
[0.40]	[1.583762507457458]	[1.583649395282541]	[0.0000113112174917]
[0.50]	[1.797629372897882]	[1.797442541400256]	[0.0000186831497625]
[0.60]	[2.044522518343920]	[2.044237600781018]	[0.0000284917562902]
[0.70]	[2.327916767873288]	[2.327505414940953]	[0.0000411352932335]
[0.80]	[2.651652651625633]	[2.651081856984936]	[0.0000570794640697]
[0.90]	[3.019974941573680]	[3.019206222313899]	[0.0000768719259780]

[1.00] [3.437575266202459] [3.436563656918091] [0.0001011609284368]

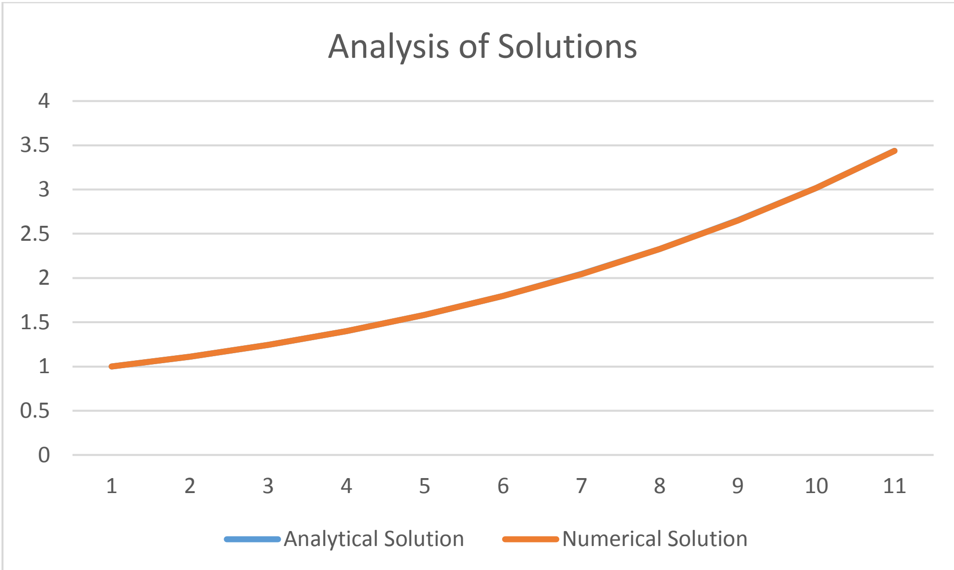

Figure 6: Analysis of Numerical and Analytical Solutions of $y' = x + y$ on line graph

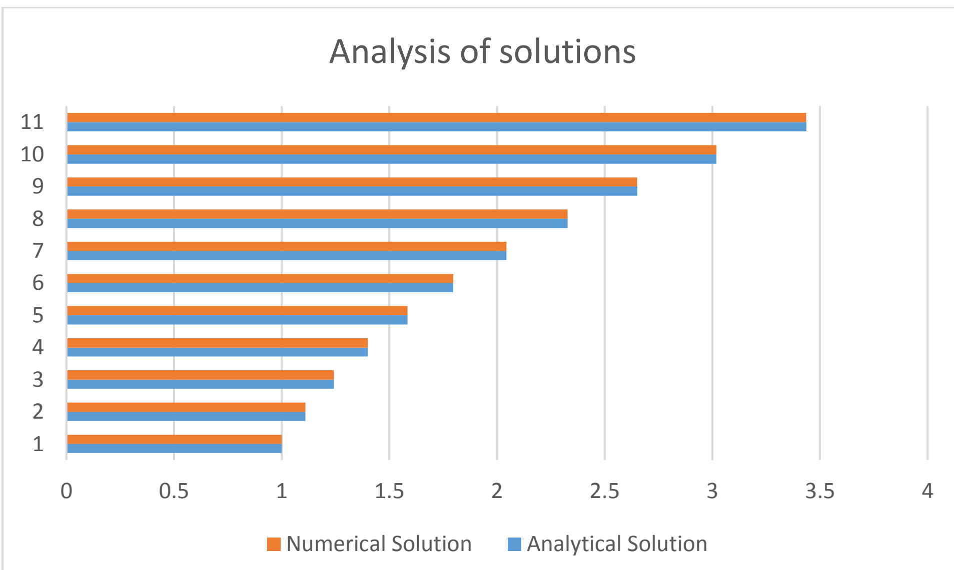

Figure 7: Analysis of Numerical and Analytical Solutions of $y' = x + y$ on bar graph

5.0 DISCUSSION AND CONCLUSION

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Numerical analysis is an area of mathematics and computer that creates, analyses, and implements algorithms for obtaining numerical solutions to problems involving continuous variables. Such Problems arise throughout the natural sciences, social sciences, engineering, medicine, and business. Computational Algorithm is used to approximate the area under the graph of a function f;

the number of step size is increased at interval to have a better approximation of the area. The effectiveness of the method can be compared with the analytical results of the test equations by considering the numerical approximations and their associated errors.

The Implementation of Scheme (14) on the problems under investigation shows that it is consistence and stable, thereby showing a measure of convergence towards the exact solution. Hence, it is reliable enough to solve Initial value problem of First Order ODEs and therefore, recommended.

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