Academic Journal of Statistics and Mathematics (AJSM)
Vol.6, No.6; June - 2020;
ISSN (5730 – 7151);
p –ISSN 4052 – 392X
Impact factor: 3.92

TWO STEP FINITE DIFFERENCE SCHEME FOR THE NUMERICAL SOLUTION OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS ARISING FROM DYNAMICAL SYSTEMS

[1]Adesoji Abraham OBAYOMI, [1]Michael Olufemi OKE, [1]Samuel Olukayode AYINDE, [2]Adewale Emmanuel ADENIPEKUN and [3]Olu Sunday ADETOLAJU.

[1,] Department of Mathematics, Ekiti State University, Ado – Ekiti, Nigeria.

[2]Department of Statistics, Federal Polytechnic, Ede, Osun State, Nigeria.

[3]Department of Computer Science, Ekiti State University, Ado – Ekiti, Nigeria.

ABSTRACT: *This paper presents a new set of two step finite difference scheme for the numerical solution of some initial value second order ordinary differential equations arising from dynamical systems. We applied a combination of non-standard transformation of the differential components and an interpolation function to create a new simulation model that can be used to approximate the dynamics of a physical phenomenon whose state equations can be represented by second order ordinary differential equations. The resulting scheme have been applied to some initial value problems and has been shown to be very suitable to a class of second order ordinary differential equations with vanishing velocity components. The schemes have been found to possess desirable qualitative properties and it converges to the analytical solution.*

Keywords: *Non-standard method, Hybrid, Interpolation function, Standard finite difference method, Dynamical model*

1. Introduction

The mathematical formulation of several physical phenomena results in non-linear ordinary differential equations, Anguelov and Lubuma (2003). Researchers have attempted to solve such equations by replacing the non-linear equations with related linear equations. This generally approximates the original equations in such a way that it produces solutions that are close enough to the dynamics of the behaviour of the original phenomena, Lambert (1991). Given enough information on the dynamics of these equations, a non-standard finite difference method can be used to develop a scheme to model the equation. Such technique seeks to produce discrete models that replicate the behavioural properties of the equation under study.Standard finite difference methods have been found to be more valuable in finding solutions at close ranges and around special grid points. However, looking holistically at the nature of the solution curves and behavioural patterns of the schemes, we discovered that most of these standard algorithms produce solution curves that do not carry along the qualitative properties of the original dynamic equations, Obayomi and Oke (2015) and Mickens (2000).In this paper, we are introducing an interpolating function and using a non-standard technique to construct finite difference schemes that will be suitable for the numerical approximation of some initial value second order ordinary differential equations. The non-standard method will help to reduce the possibility of numerical instability. Since the only way for a modeler to elicit information about a particular phenomenon is to perform experiments, we have applied the developed discrete model to some initial value problems represented by second order ordinary differential equations arising from dynamical systems.

2. Derivation of the Schemes

Let us assume an initial value problem of the form

$$y'' = f(x,y,y')\ ,\ y(x_0) = \theta \qquad (1)$$

A non-standard model for a second derivative central difference scheme may be written as

$$y'' \equiv \frac{y_{k+1} - 2y_k + y_{k-1}}{\varphi} \qquad (2)$$

where $\varphi(h) \rightarrow h^2 + 0(h^4)\ as\ h \rightarrow 0$, Obayomi and Oke (2016) and Mickens (1994).

Let us assume a solution of equation (1) that can be represented by a combination of polynomial function and a simple exponential component in the form:

$$y(x) = a_o + a_1 x + a_2 e^{-\alpha x} \qquad (3)$$

Then at points $x = x_{n-1}, x = x_n$ and $x = x_{n+1}$, we have:

$$y(x_{n-1}) = a_o + a_1 x_{n-1} + a_2 e^{-\alpha x_{n-1}}$$

$$y(x_n) = a_o + a_1 x_n + a_2 e^{-\alpha x_n}$$

$$y(x_{n+1}) = a_o + a_1 x_{n+1} + a_2 e^{-\alpha x_{n+1}}$$

then it follows that:

$$y(x_{n+1}) - 2y(x_n) + y(x_{n-1}) = a_1(x_{n+1} - 2x_n + x_{n-1}) + a_2(e^{-\alpha x_{n+1}} - 2e^{-\alpha x_n} + e^{-\alpha x_{n-1}}) \qquad (4)$$

$x_{n-1} = a + (n-1)h, x_n = a + nh$ and $x_{n+1} = a + (n+1)h$.

Then

$$y(x_{n+1}) - 2y(x_n) + y(x_{n-1}) = a_1.(0) + a_2.e^{-\alpha(a+nh)}(e^{-\alpha h} - 2 + e^{-\alpha h}) \qquad (5)$$

$$y(x) = a_o + a_1 x + a_2 e^{-\alpha x}$$

$$y'(x) = a_1 - \alpha a_2 e^{-\alpha x} \qquad (6)$$

$$y''(x) = \alpha^2 a_2 e^{-\alpha x} \qquad (7)$$

From (3), $a_o = y(x) - a_1 x - a_2 e^{-\alpha x}$ (8)

From (6), $a_1 = y'(x) + \alpha a_2 e^{-\alpha x}$ (9)

From (7), $a_2 = \frac{y''(x)}{\alpha^2 e^{-\alpha x}}$ (10)

From (6) and (7), we have:

$$a_1 = y'(x) + \frac{y''(x)}{\alpha} \qquad (11)$$

Putting (7) and (6) in (4), we obtain

$$a_o = y(x) - xy'(x) - xy''(x)\,\alpha^{-2} \qquad (12)$$

But $a_2 = \frac{y''(x)}{\alpha^2 e^{-\alpha x}} = \frac{y''(x)}{\alpha^2 e^{-\alpha(a+nh)}}$

$$y(x_{n+1}) - 2y(x_n) + y(x_{n-1}) = a_2.e^{-\alpha(a+nh)}(e^{-\alpha h} - 2 + e^{-\alpha h})$$

$$= \frac{y''(x)}{\alpha^2 e^{-\alpha(a+nh)}}.e^{-\alpha(a+nh)}(e^{-\alpha h} - 2 + e^{-\alpha h})$$

$$y(x_{n+1}) - 2y(x_n) + y(x_{n-1}) = \frac{y''(x)}{\alpha^2}.(e^{-\alpha h} - 2 + e^{-\alpha h}) \qquad (13)$$

The Interpolating function must coincide with the theoretical solution at $x = x_{n-1}, x = x_n$ and $x = x_{n+1}$ such that

$$y(x_{n+1}) - 2y(x_n) + y(x_{n+1}) = y_{n+1} - 2y_n + y_{n-1}$$

$$y_{n+1} - 2y_n + y_{n-1} = \frac{y''(x)}{\alpha^2}.(e^{-\alpha h} - 2 + e^{-\alpha h}) \qquad (14)$$

The above relationship is true for second order equation whose solution can be approximated with the interpolating function.

The renormalized scheme can therefore be written in line with non-standard rules as

$$y_{n+1} - 2y_n + y_{n-1} = \varphi.\frac{y''(x)}{\alpha^2}.(e^{-\alpha h} - 2 + e^{-\alpha h}) \qquad (15)$$

Hence we have a class of schemes that can be used for approximating second order ordinary differential equations. For the purpose of testing, we may choose $\varphi \in [0,1]$.

This will be renormalized. Applying rule 2 of the non-standard modeling rules, we will obtain two new schemes by replacing h with a dynamic function of h as follows $\psi(h) \to h + 0(h^2)\ as\ h \to 0$.

$\psi = \sin(h)$, $\psi = \frac{(e^{\lambda h}-1)}{\lambda}$, $\psi = \sin(\propto h), \psi = h$

$\propto, \lambda \in \mathbb{R}$

The scheme developed in (15) will be named NEW h

The hybrid scheme obtained by substituting h for $\psi = \sin(h)$ and $\psi = \frac{(e^{\lambda h}-1)}{\lambda}$ will be named NEW Sin and NEW Exp respectively.

3. Qualitative Properties of the New Scheme

Theorem

Let $y_n = y(x_n)$ and $p_n = p(x_n)$ denote two different numerical solutions of the differential equation with the initial conditions specified as $y_0 = y(x_0) = \xi$ and $p_0 = p(x_0) = \xi^*$ respectively such that $|\xi - \xi^*| < \varepsilon$, for $\varepsilon > 0$ If the two numerical estimates are generated by the integration schemes, we have

$$y_{n+1} = y_n + h\phi(x_n, y_n, h)$$

$$p_{n+1} = p_n + h\phi(x_n, p_n, h)$$

The condition that

$$|y_{n+1} - p_{n+1}| \le K\,|\xi - \xi^*|$$

is the necessary and sufficient condition for the stability and convergence of the schemes, Fatunla (1988)

Proof of Convergence

$$y_{n+1} = 2y_n + y_{n-1} + \varphi.\frac{y''(x)}{\alpha^2}.(e^{-\propto h} - 2 + e^{-\propto h})$$

$$y_{n+1} = 2y_n - y_{n-1} + \varphi\left(\frac{f_n'}{\propto^2}\right)(e^{\propto h} - 2 + e^{-\propto h}) \quad (16)$$

Let

$$A = \frac{\varphi(e^{\propto h} - 2 + e^{-\propto h})}{\propto^2}$$

$$y_{n+1} = 2y_n - y_{n-1} + Af_n' \quad (17)$$

For small h, $2y_n - y_{n-1} \cong y_n$

Simplify to obtain

$$y_{n+1} = y_n + Af_n'$$

The incremental function can be written as

$$\phi(x_n, y_n, h) = Af_n' \quad (18)$$

$$\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h) = A[f'(x_n, y_n, h) - f'(x_n, y_n^*, h)]$$

$$= A[f'(x_n, y_n) - f'(x_n, y_n^*)]$$

$$= A[\frac{\partial f'(x_n, \bar{y})}{\partial y_n}(y_n - y_n^*)]$$

$$L = \text{SUP}_{(x_n, y_n) \in D}\ \frac{\partial f'(x_n, \bar{y})}{\partial y_n}$$

then

$$\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h)$$

$$= A[L(y_n - y_n^*)] \quad (19)$$

$$\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h) \le L|y_n - y_n^*|$$

Equation (19) is the condition for convergence

Consistency of the schemes

$$y_{n+1} = y_n + Af_n' \quad (20)$$

Then

$$y_{n+1} = y_n + h\,\phi(x_n, y_n, h)$$

When $h = 0$, $(e^{\propto h} - 2 + e^{-\propto h}) = 0$

$\Rightarrow y_{n+1} = y_n$ and the incremental function is identically zero when $h = 0$

$\Rightarrow (x_n, y_n, 0) \equiv 0$, Henrici (1962).

Academic Journal of Statistics and Mathematics (AJSM)
Vol.6, No.6; June - 2020;
ISSN (5730 – 7151);
p –ISSN 4052 – 392X
Impact factor: 3.92

Stability of the schemes

Consider the equation

$$y_{n+1} = y_n + \{A\} f_n'(x_n, y_n) \quad (21)$$

Let $p_{n+1} = p_n + A\{N\} f_n'(x_n, P_n)$

$$y_{n+1} - p_{n+1} = y_n - p_n + \{A\}[f_n'(x_n, y_n) - f_n'(x_n, P_n)]$$

$$= y_n - p_n + A\left[\frac{\partial f'(x_n, p_n)}{\partial p_n}(y_n - p_n)\right]$$

L1 = $SUP_{(x_n, y_n)\in D} \frac{\partial f'(x_n, p_n)}{\partial p_n}$

$$y_{n+1} - p_{n+1} = y_n - p_n + A.L1(y_n - p_n) \quad (22)$$

$$|y_{n+1} - p_{n+1}| = |y_n - p_n| + [A.L1]|(y_n - p_n)|$$

Let L = $|1+[A.L1]|$

$$|y_{n+1} - p_{n+1}| \le L\,|y_n - p_n| \quad (23)$$

Let $y_0 = y(x_0) = \xi$ and $p_0 = p(x_0) = \xi^*$ then

$$|y_{n+1} - p_{n+1}| \le K\,|\xi - \xi^*| \quad (24)$$

4. **Application to some Second Order Initial Value Problems** For the purpose of testing, we have selected a class of second order initial value problems in which the velocity components or first derivatives are absent. The resulting scheme does not have a separate velocity component.

Problem 1

$$y'' = 16 + 64y,$$
$$y(0) = 1, y'(0) = 0 \quad (25)$$

Using

$$y_{n+1} = 2y_n + y_{n-1} + \phi.\frac{y''(x)}{\alpha^2}.(e^{-\alpha h} - 2 + e^{-\alpha h})$$

The standard scheme is

$$y_{n+1} = 2y_n + y_{n-1} + \phi\frac{(16+64y)}{\alpha^2}.(e^{-\alpha h} - 2 + e^{-\alpha h}) \quad (26)$$

The Analytic solution is $y = \frac{5}{8}e^{8x} + \frac{5}{8}e^{-8x} - \frac{1}{4}$, Zill and Cullen (2005)

The non-standard scheme that does not involve interpolating function can be obtained using rules 2 and 3, thus

$$y'' = 16 + 64y$$

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{\varphi} = 16 + 64y_k$$

where

$$\varphi(h) \to h^2 + 0(h^4)\ as\ h \to 0 \quad (27)$$

$$y_{k+1} = 2y_k + y_{k-1} + \varphi(h)(16 + 64y_k) \quad (28)$$

Problem 2

$$y'' = y, \ y(0) = 1, y'(0) = 1 \quad (29)$$

Using

$$y_{n+1} = 2y_n + y_{n-1} + \phi.\frac{y''(x)}{\alpha^2}.(e^{-\alpha h} - 2 + e^{-\alpha h})$$

The standard scheme is

$$y_{n+1} = 2y_n + y_{n-1} + \phi\frac{(y_n)}{\alpha^2}.(e^{-\alpha h} - 2 + e^{-\alpha h})$$

(30)

The Analytic solution is $y = e^x$, Zill and Cullen (2005)

The non-standard scheme that does not involve interpolating function can be obtained using rules 2 and 3, thus

$$y'' = y,$$

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{\varphi} = y_k$$

where $\varphi(h) \to h^2 + 0(h^4)\ as\ h \to 0$

$$y_{k+1} = 2y_k + y_{k-1} + \varphi(h)(y_k) \quad (31)$$

Problem 3

$$y'' = -2 - 4y, \ y\left(\frac{\pi}{8}\right) = \frac{1}{2} \quad (32)$$

Using

$y_{n+1} = 2y_n + y_{n-1} + \phi . \frac{y''(x)}{\alpha^2} . (e^{-\propto h} - 2 + e^{-\propto h})$

The standard scheme is

$y_{n+1} = 2y_n + y_{n-1} + \phi \frac{(-2-4y)}{\alpha^2} . (e^{-\propto h} - 2 + e^{-\propto h})$ (33)

The Analytic solution is $y = \sqrt{2}\, sin2x - \frac{1}{2}$, Zill and Cullen (2005)

The non-standard scheme that does not involve interpolating function can be obtained using rules 2 and 3, thus

$y'' = -2 - 4y$

$\frac{y_{k+1} - 2y_k + y_{k-1}}{\varphi} = -2 - 4y_k$

where $\varphi(h) \rightarrow h^2 + 0(h^4)\ as\ h \rightarrow 0$ $y_{k+1} = 2y_k + y_{k-1} + \varphi(h)(-2 - 4y_k)$ (34)

5. Testing and Experimentation

The schemes have been tested using various step sizes and the behaviours of the curves were consistent. We present below the 3D graphs for the scheme using step size $h = 0.01$

Problem 1 Schemes of $y'' = 16 + 64y,\ y(0) = 1,\ y'(0) = 0$

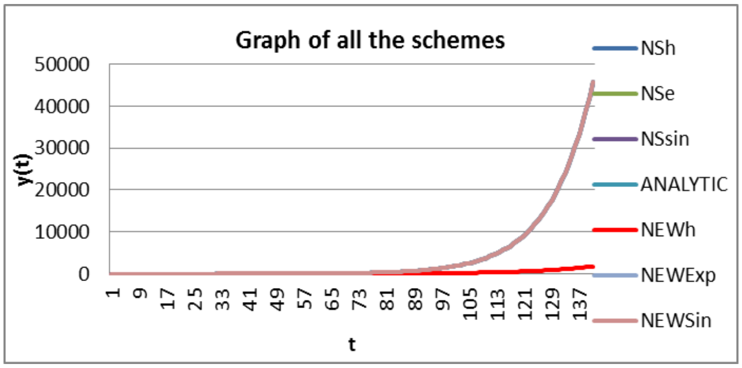

Fig 1: Solution Curves for the Standard, Hybrid and Non-standard Schemes of problem 1

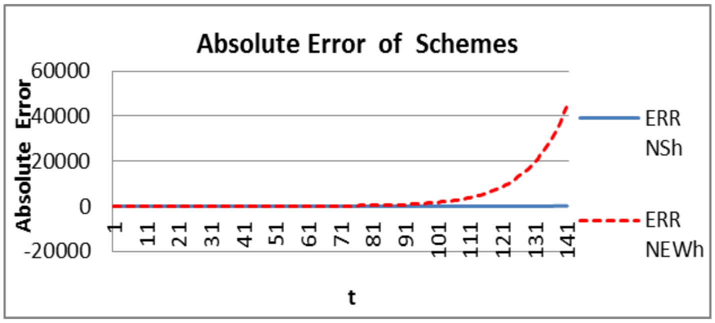

Fig 2a: Graph of Absolute Error for the Hybrid and Non-standard Schemes of Problem 1 (NSh and NEW h)

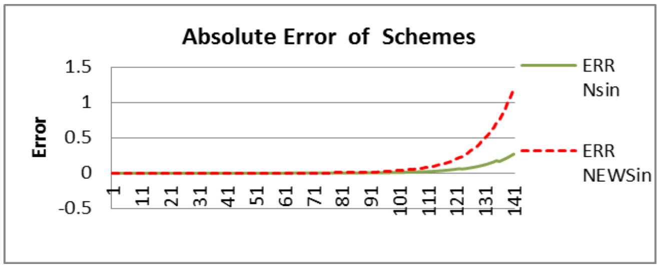

Fig 2b: Graph of Absolute Error for the Hybrid and Non-standard Schemes of Problem 1(NSin and NEW Sin)

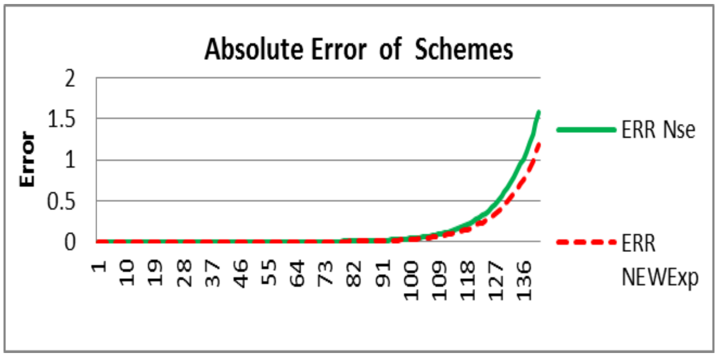

Fig 2c: Graph of Absolute Error for the Hybrid and Non-standard Schemes of Problem 1 (NSe and NEW Exp)

Problem 2 Schemes of $y'' = y,\ y(0) = 1, y'(0) = 1$

Academic Journal of Statistics and Mathematics (AJSM)
Vol.6, No.6; June - 2020;
ISSN (5730 – 7151);
p –ISSN 4052 – 392X
Impact factor: 3.92

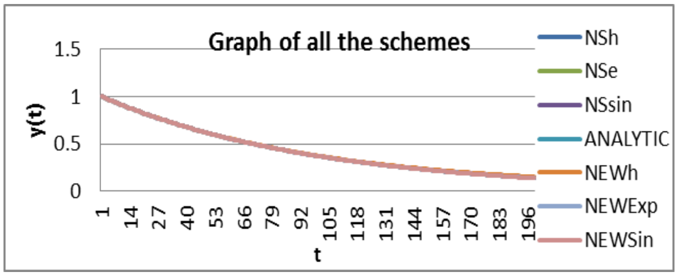

Fig 3: Solution Curves for the Standard, Hybrid and Non-standard Schemes of Problem 2

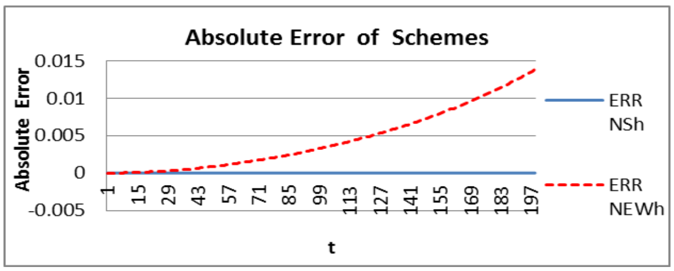

Fig 4a: Graph of Absolute Error for the Hybrid and Non-standard Schemes of Problem 2 (NSh and NEW h)

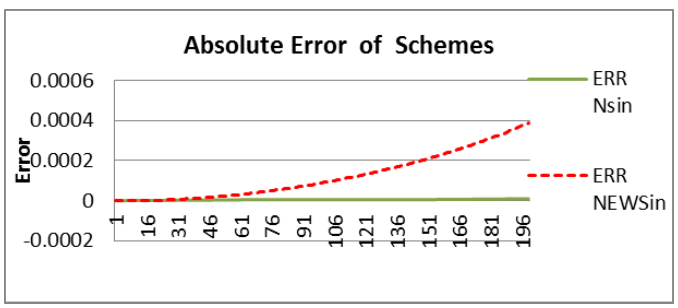

Fig 4b: Graph of Absolute Error for the Hybrid and Non-standard Schemes of Problem 2 (NSin and NEW Sin)

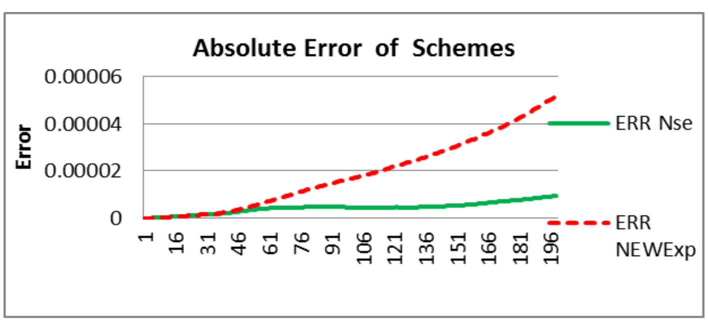

Fig 4c: Graph of Absolute Error for the Hybrid and Non-standard Schemes of Problem 2 (NSe and NEW Exp)

Problem 3 Schemes of $y'' = -2 - 4y,\ y\left(\frac{\pi}{8}\right) = \frac{1}{2}$

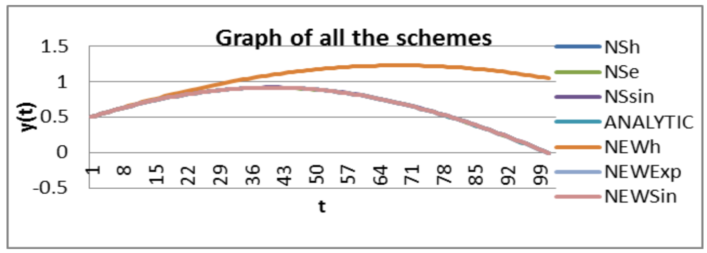

Fig 5: Solution Curves for the Standard, Hybrid and Non-standard Schemes of Problem 3

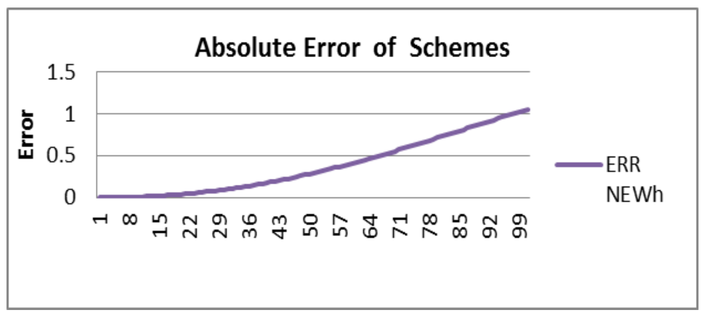

Fig 6a:Graph of Absolute Error for the Hybrid and Non-standard Schemes of Problem 3(NEW h)

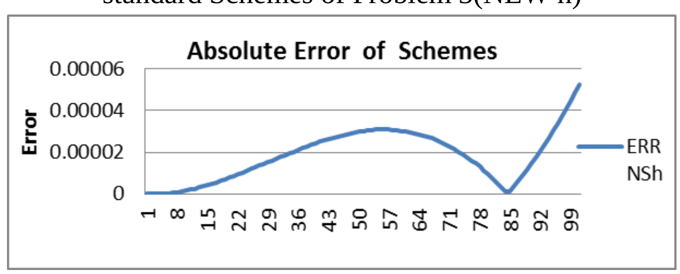

Fig 6b: Graph of Absolute Error for the Hybrid and Non-standard Schemes of Problem 3 (NSh)

Academic Journal of Statistics and Mathematics (AJSM)
Vol.6, No.6; June - 2020;
ISSN (5730 – 7151);
p –ISSN 4052 – 392X
Impact factor: 3.92

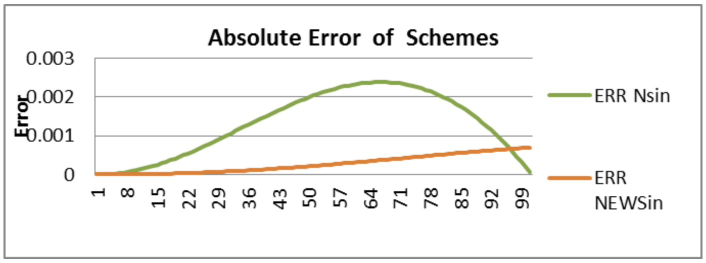

Fig 6c: Graph of Absolute Error for the Hybrid and Non-standard Schemes of Problem 3 (NSin and NEW Sin)

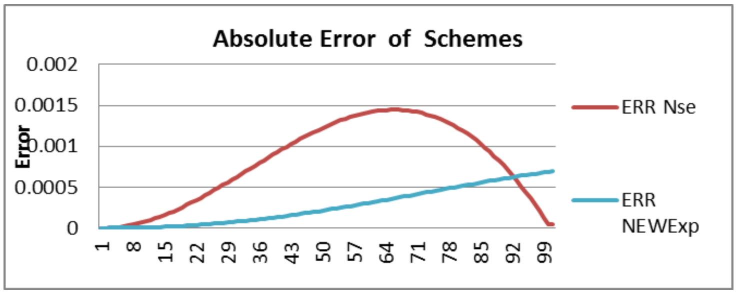

Fig 6d: Graph of Absolute Error for the Hybrid and Non-standard Schemes of Problem 3 (NSe and NEW Exp)

6. Discussion of Results and Conclusion

The schemes have been tested on second order ordinary differential equation with a vanishing velocity component. We observed that all the schemes obey the monotonic properties of the dynamics of the original equation as we can see in Figures 1, 3 and 5. It can be observed that the standard scheme NEW h possesses the highest absolute error of deviation from the analytic solution as we can see in figures 2a, 4a and 6a. There is a lot of improvement on the error of deviation for the hybrid schemes NEW Sin and NEW Exp. All the schemes have been found to be consistent. In this work, we have come up with an hybrid of standard and non-standard schemes that can be used to simulate the solution of a class of second order ordinary differential equations. The non-standard schemes performed better in the numerical experiment when the error of deviation from the analytic solution is considered.

References

[1]. Anguelov, R. and Lubuma, J.M.S. (2003): Nonstandard Finite Difference Method by Nonlocal Approximation, Mathematics and Computers in Simulation, 6, 465 - 475.

[2]. Fatunla, S. O. (1988): Numerical Methods for Initial Values Problems in Ordinary Differential Equations, Academic Press, New York.

[3]. Henrici, P. (1962): Discrete Variable Methods in Ordinary Differential Equations, John Willey & Sons, New York.

[4]. Lambert, J. D. (1991): Numerical Methods for Ordinary Differential System, John Willey & Sons, New York.

[5]. Mickens, R.E. (1994): Nonstandard Finite Difference Models of Differential Equations, World Scientific, Singapore.

[6]. Mickens, R.E. (2000): Applications of Nonstandard Methods for Initial Value Problems, World Scientific, Singapore.

[7]. Obayomi, A.A and Oke, M.O. (2015): A Non-Standard Numerical Approach to the Solution of some Second-order Ordinary differential Equations, Asian-European Journal of Mathematics, 8(4), 1 – 7, Singapore.

[8]. Obayomi, A.A and Oke, M.O. (2016): Derivation and Implementation of Two-step Finite Difference Schemes for Non-Autonomous Second Order Ordinary Differential Equations, Journal of the Nigerian Association of Mathematical Physics, 36, 65 – 72.

[9]. Zill, D.G. and Cullen, R.M. (2005): Differential Equations with Boundary Value Problems, Brooks/Cole Thompson Learning Inc., Canada.