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MASS TRANSFER AND CHEMICAL REACTION EFFECTS ON UNSTEADY MHD FLOW OF NON-NEWTONIAN FLUID WITH PRESSURE GRADIENT

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Abstract: This study investigates the mass transfer and chemical reaction effects on unsteady MHD flow nonNewtonian fluid with pressure gradient. Here, both the upper plate and the bottom plate are stationary while the pressure gradient is varying, since the flow is poiseuille. Solutions for momentum and concentration equations are obtained by the He-Laplace scheme. The effect of various flow parameters controlling the physical situation is discussed with the aid of graphs. Significant results from this study, show that both the velocity and concentration fields decrease in magnitude with the increase in Schmidt number and chemical reaction parameter. Furthermore, the velocity of the fluid increases significantly with increase in pressure gradient parameter. The results of this work are applicable to industrial processes such polymer extrusion of dye, draining of plastic films etc.

Keywords: MHD, Chemical reaction, Fourth-grade fluid, He-Laplace Scheme

Nomenclature

B_0 – external magnetic field
C – species concentration
u – fluid velocity
C_p – specific heat capacity
Ha – Hartmann number
G_c – Grashof number due to mass transfer
K_r – chemical reaction parameter
S_c –Schmidt number
C_w –concentration at the surface
C_∞ – concentration as $y \to \infty$
x, y – cartesian coordinates

Greek Symbols

μ – coefficient of shear viscosity
α – second grade parameter
β_a, β_b – third grade parameters

γ_a, γ_b – fourth grade parameters
β_c – concentration expan\sion coefficient
σ – Stefan – Boltzmann constant
ρ – density of the fluid
ν – kinematic viscosity
λ - pressure gradient

1. Introduction

The study of mass and chemical reaction is of most realistic significance to engineers and scientists because of its universal incidence in many branches of science and engineering. This phenomenon plays a significant role in chemical industry, power and cooling industry for dying, evaporation, energy transfer in cooling tower and flow in desert cooler, etc. Satya et.al [1].

Mass transfer operations are concerned with the transfer of matter from one stream to another. In many processes a change in phase may also be involved examples are dyhydration of water, solvent extraction of oil, distillation of alcohol and other volatile components, packaging of gases and vapours etc.While, Chemical reaction is a process that involves rearrangement of the molecular or ionic structure of a substance, as distinct from a change in physical form or nuclear reaction. There are two types of such reactions namely homogeneous reaction which occurs uniformly throughout a given phase of a flow and heterogeneous reaction which takes place in a particular region or within the boundary of a phase Umavathi [2]. And Magneto hydrodynamics (MHD) is the study of the magnetic properties and behaviour of electrically conducting fluids. Example of such magneto fluids include plasma, liquid metals, salt water and electrolytes. The fundamental concept of MHD is that magnetic fields can induce currents in a moving conductive fluid, which in turn polarizes the fluid and reciprocally changes the magnetic field itself. And, couette flow is the flow of a viscous fluid in the space between two surfaces, one of which is moving tangentially relative to the other. The configuration often takes the form of two parallel plates.

The non-Newtonian fluid considered in this paper is fourth grade fluid. The fourth grade fluid flow model is an exceptional model which is being used to explain the flow attitude of non-Newtonian fluids, which are considered vital and applicable in many industrial producing processes such in the drilling of oil and gas wells, polymer extrusion from dye, glass fibre, paper production and draining of plastics films etc. A vast analysis of non-Newtonian fluids problems has been done by many researches, amongst whom are Rehan *et al* [3], Islam *et al* [4], Shezad *et al* [5], Khan *et al* [6], Santhosa *et al* [7], Mohan *et al* [8], Opanagu *et al* [9], Sumathia *et al* [10], Hayat *et al* [11], Khan *et al* [12], Sajid *et al* [13] and Famikinwa *et al* [14] . Researchers that considered the effects of thermal radiation and chemical reaction include Maglesh and Gorla [15], who investigated the unsteady flow of an electrically conducting incompressible non-Newtonian viscoelastic fluid through a porous medium filled in a vertical porous channel in the presence of transverse magnetic field. The fluid and the channel rotate as a solid body with constant angular velocity about an axis perpendicular to the planes of the plates. The effects of thermal radiation and chemical reaction are taken into account embedded with slip boundary condition. The closed-form analytical solutions are obtained for momentum, energy and concentration equations. The influences of the various parameters entering into the problem in the velocity, temperature and concentration field are discussed with the help of graphs. Also, numerical values of physical quantities, such as skin friction coefficient, Nusselt number and Sherwood number are presented in tabular form. Joshi [16], who presented an analytical solution to the problem of plane Poiseuille flow in a channel filled with a porous medium. Arifuzzaman *et al*. [17], who analysed heat and mass transfer characteristics of naturally corrective hydro-magnetic flows of fourth grade radiative fluid resulting from vertical porous plate. They considered non-linear order chemical reaction and heat generation with thermal diffusion. The complete fundamental equations were transformed into dimensionless equations by implementing finite difference scheme explicitly.

Others and recent investigations were made by Idowu and Sani [18], who carried out an analysis for unsteady magnetohydrodynamic (MHD) flow of a generalized third grade fluid between two parallel plates. The fluid flow was as a result of the plate oscillating, moving and pressure gradient. Three flow problems were investigated, namely: Couette, Poiseuille and Couette-Poiseuille flows and a number of nonlinear partial differential equations were obtained which were solved using the He-Laplace method. Expressions for the velocity field, temperature and concentration fields were given for each case and finally, effects of physical parameters on the fluid motion, temperature and concentration were plotted and discussed. They found that an increase in the thermal radiation parameter increases the temperature of the fluid and hence reduces the viscosity of the fluid while the concentration of the fluid reduces as the chemical reaction parameter increases. Priya *et al.* [19] presented an analysis to explore analytical treatment for the computation of Poiseuille flow of a micropolar fluid in a channel placed in between two horizontal parallel plates. Both the plates were placed at constant wall temperatures. Therefore, the flow region was portioned into two different zones named zone I and zone II. Eringen's micropolar fluid flow phenomena took place by assuming no-slip conditions at the interface. Suitable nondimensional variables were imposed for the transformation of governing equations. Analytical treatment was carried out by employing the in-house symbolic command using the MAPLE software. The behavior of several contributing parameters such as material parameters, the couple stresses for both the zones on the velocity, and microrotation profiles were investigated and presented via graphs. The volume flow rate was also calculated and presented in tabular form. The major outcomes of the results were presented as the higher the Reynolds number, the rate increases significantly. The profile was tiled near the central region with a pick starting from the lower plate region to the central region in zone I and retards from the central region to the upper plate in the zone II, and the profiles of angular momentum seem to be symmetric in nature about the central region that was shown in both the zones. And, Joseph *et al.* [20] investigated the unsteady MHD flow of fourth-grade fluid in horizontal parallel plates channel. The upper plate was oscillating and moving while the bottom plate remained stationary. Solutions for momentum, energy and concentration equation were obtained by the HeLaplace scheme. The effect of various flow parameters controlling the physical situation is discussed with the aid of graphs. Significant results from this study, showed that velocity and temperature fields increase with the increase in thermal radiation parameter, while the velocity and concentric fields decrease with increase in chemical reaction parameter. Furthermore, velocity, temperature and concentric fields decrease with the increase in suction parameter.

However, in the above aforementioned investigations, the effect of mass transfer and chemical reaction on the unsteady MHD flow of non-Newtonian fluid (fourth grade fluid) with pressure gradient has not been paid much attention. We shall therefore, consider this where both the lower plate and the upper plate are fixed.

2. Formulation of the Problem

We consider an unsteady flow of an electrically incompressible fourth grade fluid between two parallel plates distance h apart and subjected to a uniform transverse magnetic field B_0. Here, both plates are fixed (stationary) and with constant fluid speed u. The chemically reactive flow is heading $x -$ direction along infinite porous plate with heat generation.

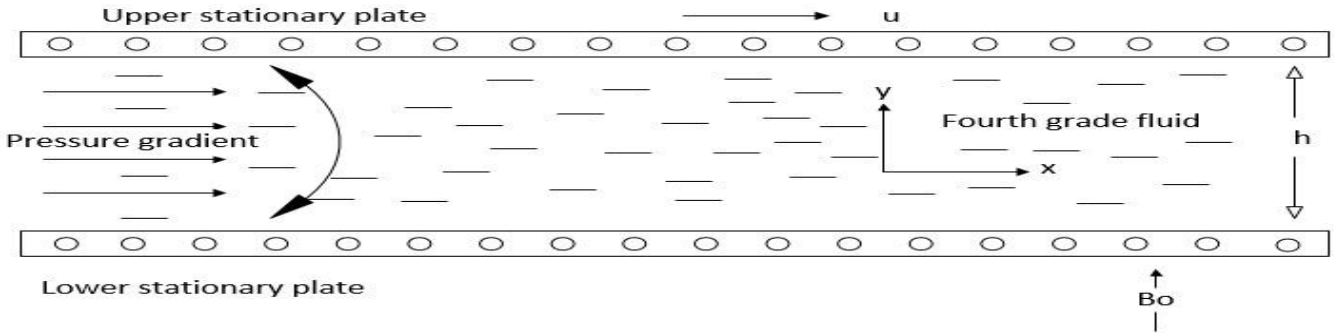

Figure 1: The geometry of the flow system

The state of this fluid is determined by the history of the deformation gradient without a preferred reference configuration. Its constitute equation can be written as

$$T(x,t) = -PI + f_{s=0}^{\infty}\left(F_t^t(s)\right) \quad (1)$$

Where PI is the undetermined part of the stress – tensor, F is the deformation gradient and f is the functional.

Coleman and Noll [21] prescribed different sorts of incompressible fluid category n as viscous fluid agreeing on Hayat et al. [22]. Incompressible fluid of differential type of grade n is the simple fluid obeying the constitutive equation

n

T □□ □^{PI} □S_j (2)

j□1

obtained by asymptotic expansion of the functional in equation (1) through a retardation parameter α. For $n = 4$ as elucidate by Hayat et al. ([22], [23]]) and Arifuzzaman et al. [17], the first four (4) tensors S_j are given by

$S_1 = \mu A_1$ (3)

$S_2 = \alpha_1 A_2 + \alpha_2 A_1^2$ (4)

$S_3 = \beta_1 A_3 + \beta_2(A_1A_2 + A_2A_1) + \beta_3(trA_1^2)A_1$ (5)

$S4 = \gamma 1A4 + \gamma 2(A3A1 + A1A3) + \gamma 3A22 + \gamma 4(A2A12 + A12A2) + \gamma 5(trA2)A2 + \gamma 6(trA2)A12 +$
$[\gamma_7 trA_3 + \gamma_8 tr(A_2A_1)]A_1$ (6)

Where, μ is the coefficient of shear viscosity, $\alpha_i(i = 1,2)$, $\beta_i(i = 1,2,3)$ and $\gamma_i(i = 1(1)8)$ are material constants. The A_n are the Rivlin – Ericksen tensors defined by the recursion relation

$$A_n = \frac{d}{dt}A_{n-1} + A_{n-1}L + L^T A_{n-1}, \; n>1 \quad (7)$$

$A_1 = L + L^T$ (8)

where $L = \nabla V$, $\frac{d}{dt}$ is the material time derivative and V is the velocity.

The chemically reactive flow is heading x – direction along infinite porous plate with heat generation. Here, U_0 is the uniform velocity, and C_∞ is the fluid temperature and concentration.

Under the above consideration, the equations that described the physical circumstances are

(I) The Momentum equation:

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} + \frac{\alpha_1\nu}{\rho}\frac{\partial^3 u}{\partial y^2\partial t} + \frac{\beta_1\nu^2}{\rho}\frac{\partial^4 u}{\partial y^2\partial t^2} + \frac{6(\beta_2+\beta_3)}{\rho}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} + \frac{\gamma_1\nu^3}{\rho}\frac{\partial^5 u}{\partial y^2\partial t^3} + \frac{2\nu(3\gamma_2+\gamma_3+\gamma_4+\gamma_5+\gamma_7+\gamma_8)}{\rho C_p}\left[2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t\partial y} + \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial y^2\partial t}\right] - \frac{\sigma B_0^2}{\rho C_p}u + g\beta_C(C - C_\infty) - \frac{\nu}{k}u \quad (9)$$

(II) The Concentration equation:

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial y^2} - K_c(C - C_\infty) \tag{10}$$

Where u is the fluid velocity and C is the species concentration equation.
We note that, when $\gamma_i = 0$, the fourth grade model reduces to the third grade model. When $\beta_i = 0$, the third grade model reduces to second grade model. When $\alpha_i = 0$, $\beta_i = 0$ and $\gamma_i = 0$ then the model reduces to classical Navier – Stoke equation.
The initial and boundary conditions Idowu and Sani [18] are

$$u(y,t) = h + h\cos(\omega y)\ ,\ C(y,t) = h + h\cos(\omega y)\ at\ y > 0,\ t = 0$$
$$u(y,t) = 1\ ,\ C(y,t) = 1\ at\ y = 0\ and\ t \geq 0 \tag{11}$$
$$u(y,t) \to \infty,\ C(y,t) \to \infty\ as\ y \to \infty\ at\ t > 0$$

In order to transform equations (2.10) – (2.13), we use the following dimensionless parameters

$$u^* = \frac{u}{U_0},\ p^* = \frac{p}{\mu U_0^2}, t^* = \frac{tU_0^2}{\nu},, G_c = \frac{g\beta_C(C_w - C_\infty)\nu}{U_0^3}, Ha^2 = \frac{\sigma B_0^2 \nu}{\rho U_0^2}, Da = \frac{KU_0^2}{h^2}, S_c = \frac{D}{\nu}, y^* = \frac{yU_0}{\nu}, x^* = \frac{x}{h}, h = \frac{U_0}{\nu}, C^* = \frac{C - C_0}{C_w - C_\infty}, \alpha = \frac{\alpha_1 U_0^2}{\rho \nu^2}, \beta_a = \frac{\beta_1 U_0^4}{\rho \nu^3}, \beta_b = \frac{(\beta_2+\beta_3)U_0^4}{\rho \nu^3}, \gamma_a = \frac{\gamma_1 U_0^6}{\rho \nu^3}, \gamma_b = \frac{2(3\gamma_2+\gamma_3+\gamma_4+\gamma_5+3\gamma_7+\gamma_8)U_0^6}{\rho \nu^4}, Da = \frac{kU_0^2}{\nu^2}, K_r = \frac{K_c \nu}{U_0^2} \tag{12}$$

Substituting equation (12) into equations (9) – (11) and by dropping the asterisks, we have the following:

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial^3 u}{\partial y^2 \partial t} + \beta_a \frac{\partial^4 u}{\partial y^2 \partial t^2} + \beta_b\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_a \frac{\partial^5 u}{\partial y^2 \partial t^3} + \gamma_b\left[2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial y^2 \partial t}\right] - \left(Ha + \frac{1}{Da}\right)u + G_r\theta + G_c C \tag{13}$$

$$\frac{\partial C}{\partial t} = \frac{1}{S_c}\frac{\partial^2 C}{\partial y^2} - K_r C \tag{14}$$

While, the initial and boundary conditions becomes

$$\left.\begin{aligned} u(y,t) = 1 + \cos\omega y,, C(y,t) = 1 + \cos\omega y\ \ at\ t = 0\ for\ 0 \leq y \geq 1 \\ u(y,t) = 1, C(y,t) = 1\ at\ y = 0\ for\ t \geq 0 \\ u(y,t) \to \infty, C(y,t) \to \infty\ as\ y \to \infty\ for\ t > 0 \end{aligned}\right\} \tag{15}$$

3. Method of Solution/Solution of the Problem

In this section we employ the He – Laplace scheme to solve equations (13) and (14) subjects to the initial and boundary conditions (15).
Since equation (13) is a coupled non – linear partial differential equation, we have to solve equations (14) first as follows;
Now applying Laplace transform on equation (14), we have;

$$L\left\{\frac{\partial C}{\partial t}\right\} = \frac{1}{S_c}L\left\{\frac{\partial^2 C}{\partial y^2}\right\} - L\{K_r C\} \tag{16}$$

Applying the initial condition and dividing through by s and rearranging, we obtain;

$$L\{C(y,t)\} = \frac{1+\cos(\omega y)}{s} + \frac{1}{s}\left\{\frac{1}{S_c}L\left\{\frac{\partial^2 C}{\partial y^2}\right\} - L\{K_r C\}\right\} \tag{17}$$

Taking the inverse Laplace transform of both sides of equation (17), gives;

$$C(y,t) = 1 + \cos(\omega y) + L^{-1}\left[\frac{1}{s}\left\{\frac{1}{S_c}L\left\{\frac{\partial^2 C}{\partial y^2}\right\} - L\{K_r C\}\right\}\right] \tag{18}$$

Applying the Homotopy perturbation technique, equation (18) yields

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P Cn n(y t,) e y P L 1 1s S 1c L 2yC2 L K C r (19) n 0

Comparing the coefficients of the like powers of $'P'$, the following approximations were obtained;

$$P: C_1(y,t) = L\left[\frac{}{s}\left\{\frac{}{S_c} L\left\{\frac{}{\partial y^2}\right\} - L\{K_r C_0\}\right\}\right] = L\left\{\frac{}{S_c}\left(\frac{}{s^2}\right) - K_r\left(\frac{1+\cos(\omega y)}{s^2}\right)\right\} = \left(\frac{-\omega^2}{S_c}\cos(\omega y) - K_r(1+\cos(\omega y))\right)t \quad (21)$$

$$P^2: C_2(y,t) = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{S_c} L\left\{\frac{\partial^2 C_1}{\partial y^2}\right\} - L\{K_r C_1\}\right\}\right] = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{S_c} L\left\{\left(\frac{\omega^4}{S_c}\cos(\omega y) +\right.\right.\right.\right.$$

$$P^0: C_0(y,t) = 1 + \cos(\omega y) \quad (20)$$

1 −1 1 1 $\partial^2 C_0$ −1 1 $\omega^2\cos(\omega y)$

$$\left.\left.\left.\left.\omega^2 K_r\cos(\omega y)\right)t\right\} + L\left\{K_r\left(\frac{\omega^2}{S_c}\cos(\omega y) + K_r^2(1+\cos(\omega y)\right)t\right\}\right\}\right] =$$

$$\left(\frac{\omega^4}{S_c^2}\cos(\omega y) + \frac{2Se^{-y}}{S_c} - \frac{2K_r\omega^2}{S_c}\cos(wy) + K_r^2\cos(\omega y) + K_r^2\right)\frac{t^2}{2!} \quad (22)$$

$$P^3: C_3(y,t) = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{S_c} L\left\{\frac{\partial^2 C_2}{\partial y^2}\right\} - L\{K_r C_2\}\right\}\right] = L^{-1}\left[\frac{1}{s}\left\{L\left\{\left(\frac{-\omega^6}{S_c^3}\cos(\omega y) - \frac{3K_r\omega^4}{S_c}\cos(\omega y) - \frac{3K_r^2\omega^2}{S_c}\cos(\omega y) - K_r^3\cos(\omega y) - K_r^3\right)\frac{t^2}{2!}\right\}\right\}\right] =$$

$$\left(\frac{-\omega^6}{S_c^3}\cos(\omega y) - \frac{3K_r\omega^4}{S_c}\cos(\omega y) - \frac{3K_r^2\omega^2}{S_c}\cos(\omega y) - K_r^3\cos(\omega y) - K_r^3\right)\frac{t^3}{3!} \quad (23)$$

$$P^4: C_4(y,t) = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{S_c} L\left\{\frac{\partial^2 C_3}{\partial y^2}\right\} - L\{K_r C_3\}\right\}\right] = L^{-1}\left[\frac{1}{s}\left\{L\left\{\left(\frac{\omega^8}{S_c^4}\cos(\omega y) + \frac{4K_r\omega^6}{S_c^3}\cos(\omega y) + \frac{4K_r^2\omega^4}{S_c^2}\cos(\omega y) + \frac{4K_r\omega^2}{S_c}\cos(\omega y) + K_r^4\cos(\omega y) + K_r^4\right)\frac{t^3}{3!}\right\}\right\}\right] =$$

$$\left(\frac{\omega^8}{S_c^4}\cos(\omega y) + \frac{4K_r\omega^6}{S_c^3}\cos(\omega y) + \frac{4K_r^2\omega^4}{S_c^2}\cos(\omega y) + \frac{4K_r\omega^2}{S_c}\cos(\omega y) + K_r^4\cos(\omega y) + K_r^4\right)\frac{t^4}{4!} \quad (24)$$

Therefore, in view of equations (20), (21), (22), (23) and (24) the solution to equation (14) is,

$$C(y,t) = C_0(y,t) + C_1(y,t) + C_2(y,t) + C_3(y,t) + C_4(y,t)\cdots$$

$$C(y,t) = 1 + \cos(\omega y) + \left(\frac{-\omega^2}{S_c}\cos(\omega y) - K_r(1+\cos(\omega y))\right)t + \left(\frac{\omega^4}{S_c^2}\cos(\omega y) + \frac{2Se^{-y}}{S_c} - \frac{2K_r\omega^2}{S_c}\cos(wy) + K_r^2\cos(\omega y) + K_r^2\right)\frac{t^2}{2!} + \left(\frac{-\omega^6}{S_c^3}\cos(\omega y) - \frac{3K_r\omega^4}{S_c}\cos(\omega y) - \frac{3K_r^2\omega^2}{S_c}\cos(\omega y) - K_r^3\cos(\omega y) - K_r^3\right)\frac{t^3}{3!} + \left(\frac{\omega^8}{S_c^4}\cos(\omega y) + \frac{4K_r\omega^6}{S_c^3}\cos(\omega y) + \frac{4K_r^2\omega^4}{S_c^2}\cos(\omega y) + \frac{4K_r\omega^2}{S_c}\cos(\omega y) + K_r^4\cos(\omega y) + K_r^4\right)\frac{t^4}{4!} + \cdots \quad (25)$$

Finally, we now solve equation (13), which is rearranged to give

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial^3 u}{\partial y^2 \partial t} + \beta_a \frac{\partial^4 u}{\partial y^2 \partial t^2} + \beta_b\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_a \frac{\partial^5 u}{\partial y^2 \partial t^3} + \gamma_b\left[2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial y^2 \partial t}\right] - l_2 u + G_c C$$ where, $Ha + \frac{1}{Da} = l_2$

Applying the Laplace transform on both sides of equation (13) gives

$$L\left\{\frac{\partial u}{\partial t}\right\} = L\left\{-\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial^3 u}{\partial y^2 \partial t} + \beta_a \frac{\partial^4 u}{\partial y^2 \partial t^2} + \beta_b\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_a \frac{\partial^5 u}{\partial y^2 \partial t^3} + \gamma_b\left[2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial y^2 \partial t}\right] - l_2 u + G_c C\right\} \quad (26)$$

But, $L\left\{\frac{\partial u}{\partial t}\right\} = sL\{u(y,t)\} - u(y,0)$ (27)

Hence,

$$L\{u(y,t)\} = \frac{u(y,0)}{s} + \frac{1}{s} L\left\{-\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial^3 u}{\partial y^2 \partial t} + \beta_a \frac{\partial^4 u}{\partial y^2 \partial t^2} + \beta_b\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_a \frac{\partial^5 u}{\partial y^2 \partial t^3} + \gamma_b\left[2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial y^2 \partial t}\right] - l_2 u + G_c C\right\} \quad (28)$$

Taking the inverse Laplace transform of both sides of equation (28), we have;

$$L^{-1}\{L\{u(y,t)\}\} = L^{-1}\left\{\frac{u(y,0)}{s} - \frac{\partial p}{\partial x} + \frac{1}{s} L\left\{\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial^3 u}{\partial y^2 \partial t} + \beta_a \frac{\partial^4 u}{\partial y^2 \partial t^2} + \beta_b\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_a \frac{\partial^5 u}{\partial y^2 \partial t^3} + \gamma_b\left[2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial y^2 \partial t}\right] - l_2 u + \frac{G_c}{s}\left(1 + \cos(\omega y) + \left(\frac{-\omega^2}{S_c}\cos(\omega y) - K_r(1 + \cos(\omega y))\right)t + \left(\frac{\omega^4}{S_c^2}\cos(\omega y) + \frac{2Se^{-y}}{S_c} - \frac{2K_r\omega^2}{S_c}\cos(wy) + K_r^2\cos(\omega y) + K_r^2\right)\frac{t^2}{2!} + \left(\frac{-\omega^6}{S_c^3}\cos(\omega y) - \frac{3K_r\omega^4}{S_c}\cos(\omega y) - \frac{3K_r^2\omega^2}{S_c}\cos(\omega y) - K_r^3\cos(\omega y) - K_r^3\right)\frac{t^3}{3!} + \left(\frac{\omega^8}{S_c^4}\cos(\omega y) + \frac{4K_r\omega^6}{S_c^3}\cos(\omega y) + \frac{4K_r^2\omega^4}{S_c^2}\cos(\omega y) + \frac{4K_r\omega^2}{S_c}\cos(\omega y) + K_r^4\cos(\omega y) + K_r^4\right)\frac{t^4}{4!}\right)\right\}\right\} \quad (29)$$

Or,

$$u(y,t) = 1 + \lambda + \cos(\omega y) + G_c(1 + \cos(\omega y)) + \left(-\frac{\omega^2}{S_c}\cos(\omega y) - K_r(1 + \cos(\omega y))\right)t + \left(+\frac{\omega^4}{S_c^2}\cos(\omega y) + \frac{2Se^{-y}}{S_c} - \frac{2K_r\omega^2}{S_c}\cos(wy) + K_r^2\cos(\omega y) + K_r^2\right)\frac{t^2}{2!} + \left(-\frac{\omega^6}{S_c^3}\cos(\omega y) - \frac{3K_r\omega^4}{S_c}\cos(\omega y) - \frac{3K_r^2\omega^2}{S_c}\cos(\omega y) - K_r^3\cos(\omega y) - K_r^3\right)\frac{t^3}{3!} + \left(\frac{\omega^8}{S_c^4}\cos(\omega y) + \frac{4K_r\omega^6}{S_c^3}\cos(\omega y) + \frac{4K_r^2\omega^4}{S_c^2}\cos(\omega y) + \frac{4K_r\omega^2}{S_c}\cos(\omega y) + K_r^4\cos(\omega y) + K_r^4\right)\frac{t^4}{4!} + L^{-1}\left\{\frac{1}{s} L\left\{\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial^3 u}{\partial y^2 \partial t} + \beta_a \frac{\partial^4 u}{\partial y^2 \partial t^2} + \beta_b\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} + \gamma_a \frac{\partial^5 u}{\partial y^2 \partial t^3} + \gamma_b\left[2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^3 u}{\partial y^2 \partial t}\right] - l_2 u\right\}\right\} \quad (30)$$

Applying the Homotopy perturbation method to equation (30), gives

□

□$P u^{n}{}_{n}(\ ,\ ) = 1 + \lambda + \cos(\omega y) + G_c(1 + \cos(\omega y)) + \left(-\frac{\omega^2}{S_c}\cos(\omega y) - K_r(1 + \cos(\omega y))\right)t +$ $y\ t$

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$n\square 0$

$$\left(\frac{\omega^4}{S_c^2}\cos(\omega y)+\frac{2Se^{-y}}{S_c}-\frac{2K_r\omega^2}{S_c}\cos(wy)+K_r^2\cos(\omega y)+K_r^2\right)\frac{t^2}{2!}+\left(-\frac{\omega^6}{S_c^3}\cos(\omega y)-\frac{3K_r\omega^4}{S_c}\cos(\omega y)-\frac{3K_r^2\omega^2}{S_c}\cos(\omega y)-K_r^3\cos(\omega y)-K_r^3\right)\frac{t^3}{3!}+\left(\frac{\omega^8}{S_c^4}\cos(\omega y)+\frac{4K_r\omega^6}{S_c^3}\cos(\omega y)+\frac{4K_r^2\omega^4}{S_c^2}\cos(\omega y)+\frac{4K_r\omega^2}{S_c}\cos(\omega y)+K_r^4\cos(\omega y)+K_r^4\right)\frac{t^4}{4!}+P\left(L^{-1}\left\{\frac{1}{s}L\left\{S\frac{\partial u}{\partial y}+\frac{\partial^2 u}{\partial y^2}+\alpha\frac{\partial^3 u}{\partial y^2\partial t}+\beta_a\frac{\partial^4 u}{\partial y^2\partial t^2}+\beta_b H_a(u_n)+\gamma_1\frac{\partial^5 u}{\partial y^2\partial t^3}+\gamma_b\left[2H_b(u_n)+H_c(u_n)\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t\partial y}+\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial y^2\partial t}\right]-l_2 u\right\}\right\}\right) \tag{31}$$

Where, $H_a(u_n)$, $H_b(u_n)$ and $H_c(u_n)$ are the He's polynomials for $\left(\frac{\partial u}{\partial y}\right)^2, \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 u}{\partial t\partial y}$ and $\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial y^2\partial t}$ respectively.
Now, comparing the like powers of P in equation (31) and equating their coefficients gives

$$P^0; u_0(y,t)=1+\lambda+\cos(\omega y)+G_c(1+\cos(\omega y))+\left(-\frac{\omega^2}{S_c}\cos(\omega y)-K_r(1+\cos(\omega y))\right)t+\left(\frac{\omega^4}{S_c^2}\cos(\omega y)+\frac{2Se^{-y}}{S_c}-\frac{2K_r\omega^2}{S_c}\cos(wy)+K_r^2\cos(\omega y)+K_r^2\right)\frac{t^2}{2!}+\left(-\frac{\omega^6}{S_c^3}\cos(\omega y)-\frac{3K_r\omega^4}{S_c}\cos(\omega y)-\frac{3K_r^2\omega^2}{S_c}\cos(\omega y)-K_r^3\cos(\omega y)-K_r^3\right)\frac{t^3}{3!}+\left(\frac{\omega^8}{S_c^4}\cos(\omega y)+\frac{4K_r\omega^6}{S_c^3}\cos(\omega y)+\frac{4K_r^2\omega^4}{S_c^2}\cos(\omega y)+\frac{4K_r\omega^2}{S_c}\cos(\omega y)+K_r^4\cos(\omega y)+K_r^4\right)\frac{t^4}{4!} \tag{32}$$

$$P^1; u_1(y,t)=L^{-1}\left\{\frac{1}{s}L\left\{S\frac{\partial u_0}{\partial y}+\frac{\partial^2 u_0}{\partial y^2}+\alpha\frac{\partial^3 u_0}{\partial y^2\partial t}+\beta_a\frac{\partial^4 u_0}{\partial y^2\partial t^2}+\beta_b(u_0')^2(u_0'')+\gamma_a\frac{\partial^5 u_0}{\partial y^2\partial t^3}+\gamma_b[2u_0'''u''_{0t}+(u_0')^2(u_0''u_{0t}')]-l_2u_0\right\}\right\} \tag{33}$$

Or,

$$u_1(y,t)=(A_{11}+\alpha A_{18}+\beta_b A_{11}+2\gamma_b A_{15}A_{16}+A_{11}A_{13}A_{17}-l_2A_{19})t+(A_{12}+\beta_b A_{12}A_{13}+\beta_b A_{11}A_{14}+2\gamma_b A_{16}^2+A_{13}A_{14}A_{17}-l_2A_{10})\frac{t^2}{2!}+(2\beta_b A_{12}A_{14}+2A_{11}A_{14}^2-l_2A_{21})\frac{t^3}{3!}-l_2A_{22}\frac{t^4}{4!}-l_2A_{23}\frac{t^5}{5!} \tag{34}$$

Therefore, the solution to equation (13) is;

$$u(y,t)=u_0(y,t)+u_1(y,t)+u_2(y,t)\cdots$$

$$u(y,t)=A_{19}+(A_{10}+A_{11}+\alpha A_{18}+\beta_b A_{11}+2\gamma_b A_{15}A_{16}+A_{11}A_{13}A_{17}-l_2A_{19})t+(A_{12}+A_{21}+\beta_b A_{12}A_{13}+\beta_b A_{11}A_{14}+2\gamma_b A_{16}^2+A_{13}A_{14}A_{17}-l_2A_{10})\frac{t^2}{2!}+(A_{22}+2\beta_b A_{12}A_{14}+2A_{11}A_{14}^2-l_2A_{21})\frac{t^3}{3!}+\left(A_{23}-l_2A_{22}\frac{t^4}{4!}\right)-l_2A_{23}\frac{t^5}{5!}+\cdots \tag{35}$$

Where,

$$A_{10} = -\frac{\omega^2}{S_c}\cos(\omega y) - K_r(1+\cos(\omega y))$$

$$A_{11} = -\omega^2\cos(\omega y) - G_c\omega^2\cos(\omega y)$$

$$A_{12} = G_cK_r\omega^2\cos(\omega y) + \frac{G_c\omega^4\cos(\omega y)}{S_c}$$

$$A_{13} = \omega^2\sin^2(\omega y) + 2G_c\omega^2\sin^2(\omega y) + G_c^2\sin^2(\omega y)$$

$$A_{14} = 2\delta G_r\omega^2\sin^2(\omega y) - \frac{2G_c\omega^4}{S_c}\sin^2(\omega y) - \frac{4G_cG_r\omega^4}{S_c}\sin^2(\omega y) - \frac{G_c^2\omega^4}{S_c}\sin^2(\omega y) - K_rG_c^2\omega^2\sin^2(\omega y) + \frac{G_c^2\omega^6}{S_c^2}\sin^2(\omega y) + \frac{2K_rG_c^2\omega^4}{S_c}\sin^2(\omega y) + G_c^2K_r^2\omega^2\sin^2(\omega y)$$

$$A_{15} = \omega^3\sin^2(\omega y) + G_c\omega^3\sin^2(\omega y)$$

$$A_{16} = -\frac{G_c\omega^5}{S_c}\sin(\omega y) - G_cK_r\omega^2\sin(\omega y)$$

$$A_{17} = G_cK_r\omega\sin(\omega y) + \frac{G_c\omega^3}{S_c}\sin(\omega y)$$

$$A_{18} = \alpha G_cK_r\omega^2\cos(\omega y) - \frac{\alpha G_c\omega^2\cos(\omega y)}{S_c}$$

$$A_{19} = 1 + \lambda + \cos(\omega y) + G_c(1+\cos(\omega y))$$

$$A_{21} = \frac{\omega^4}{S_c^2}\cos(\omega y) + \frac{2Se^{-y}}{S_c} - \frac{2K_r\omega^2}{S_c}\cos(wy) + K_r^2\cos(\omega y) + K_r^2$$

$$A_{22} = -\frac{\omega^6}{S_c^3}\cos(\omega y) - \frac{3K_r\omega^4}{S_c}\cos(\omega y) - \frac{3K_r^2\omega^2}{S_c}\cos(\omega y) - K_r^3\cos(\omega y) - K_r^3$$

$$A_{23} = \frac{\omega^8}{S_c^4}\cos(\omega y) + \frac{4K_r\omega^6}{S_c^3}\cos(\omega y) + \frac{4K_r^2\omega^4}{S_c^2}\cos(\omega y) + \frac{4K_r\omega^2}{S_c}\cos(\omega y) + K_r^4\cos(\omega y) + K_r^4$$

4. Results and Discussion

Theoretical work on mass transfer and chemical reaction effects on unsteady MHD flow of non-Newtonian fluid with pressure has been analyzed and discussed in details. The impact of chemical reaction, along with other pertinent flow parameters are plotted graphically on different flow fields. The default values for the pertinent flow parameters are taken as (Joseph et al. [20]), $\lambda = 0.30, \alpha = 0.20, \beta_a = 0.05, \beta_b = 0.05$, $\gamma_a = 0.05, \gamma_b = 0.05$, $S_c = 0.50, G_c = 5$, $P_r = 0.71, Ha = 0.30, Da = 1.00, K_r = 0.50$.

The effect of chemical reaction parameter (K_r) on velocity and concentration profiles are depicted in **Fig. 2.** and **Fig. 3.** respectively. Due to the rise of chemical reaction (K_r) from $1 \leq K_r \geq 4$, the velocity field decreases, and the concentration field also decreases. Physically, chemical reaction occurs with more disturbance which develops the molecular motion and upsurges the heat transport phenomena and as a result retards the velocity of the flow.

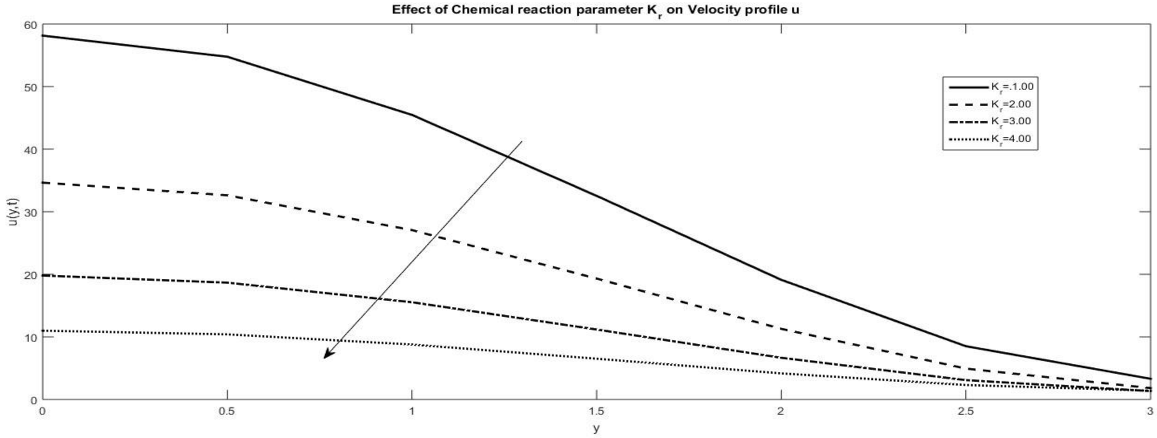

Fig. 2. Effect of chemical reaction parameter K_r on Velocity profile u

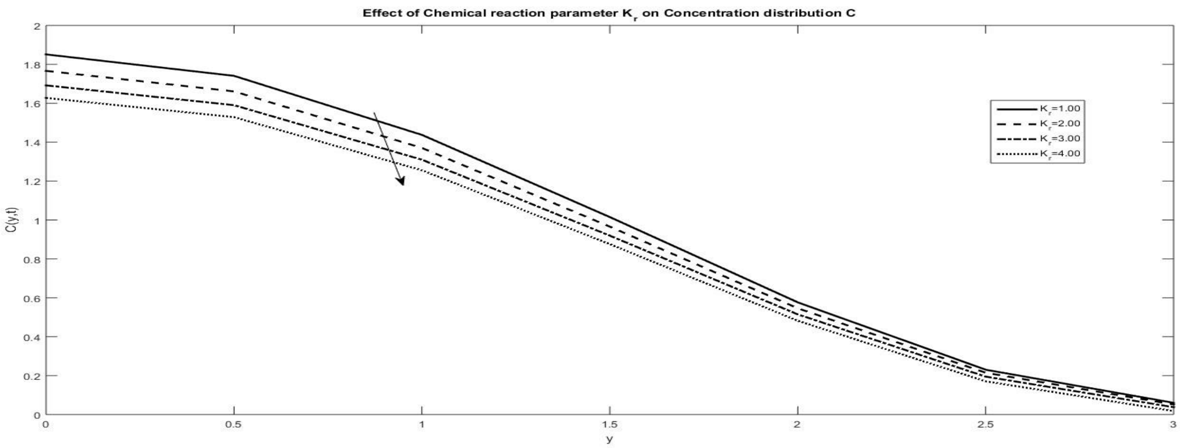

Fig. 3. Effect of chemical reaction parameter K_r on Concentration distribution C

Fig. 4. illustrates the drag force effect on fluid flow. The velocity profile decreases with the increment of Hartmann number ($1 \leq Ha \geq 4$). The role of Hartmann number which is the magnetic parameter is to suppress turbulence. Physically, when magnetic field is applied to any fluid, the apparent viscosity of the fluid increases to the point of becoming viscous elastic solid. It is of great interest that yield stress of the fluid can be controlled very accurately through variation of the magnetic field intensity. The result is that the ability of the fluid to transmit force can be controlled with help of electromagnet which give rise to many possible control – based applications, including MHD power generation, electromagnetic casting of metals, MHD propulsion etc.

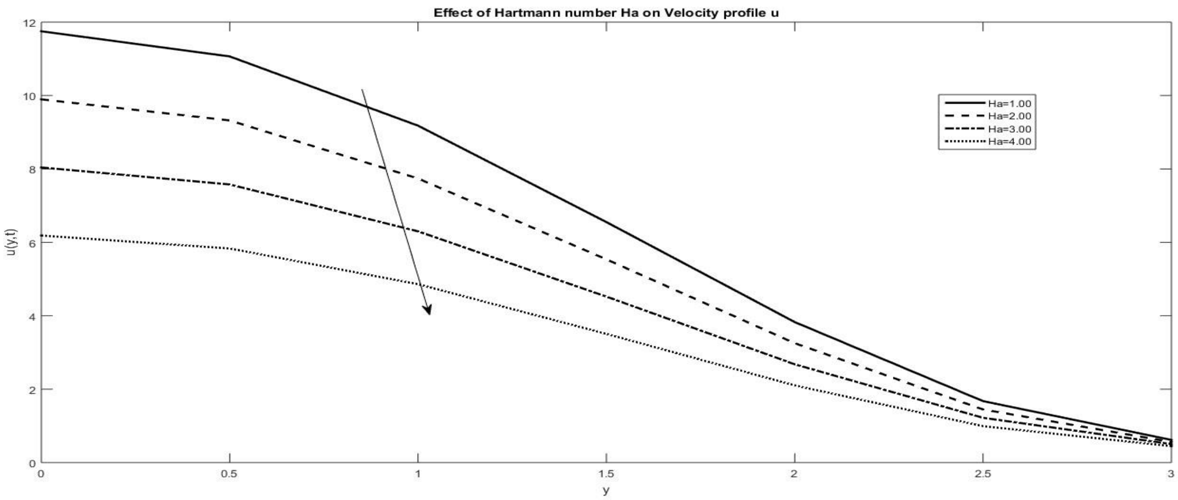

Fig. 4. Effect of Hartmann number Ha on Velocity profile u

Fig. **5.** and **Fig. 6.** display the velocity and concentration profiles respectively for the increment of Schmidt number ($1 \leq S_c \geq 4$). Both the velocity and concentration profiles decrease with increase of Schmidt number (S_c). Physically, Schmidt number (S_c) helps to develop fluid concentration and concentration buoyancy force. Furthermore, it can also be used to improve the visualization of fluid fields.

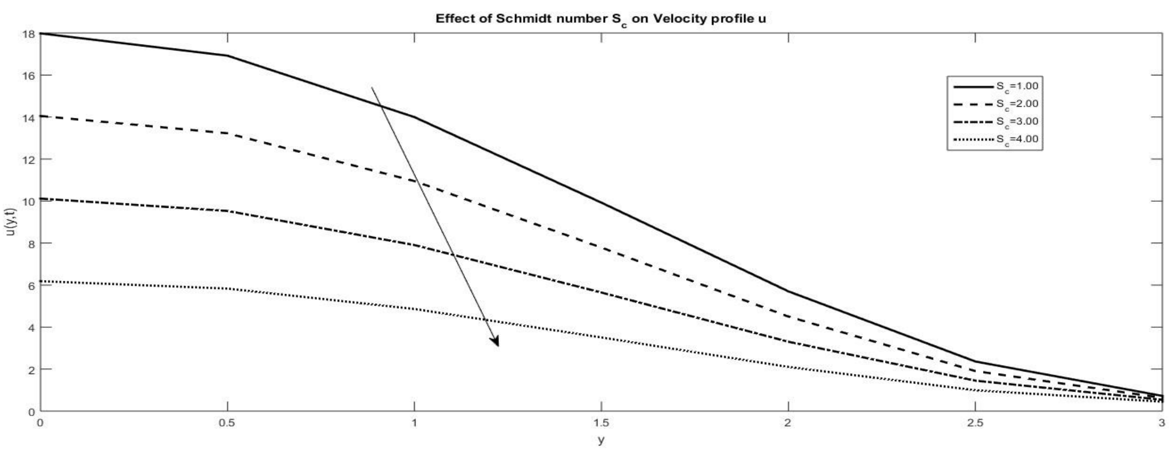

Fig. 5. Effect of Schmidt number S_c on Velocity profile u

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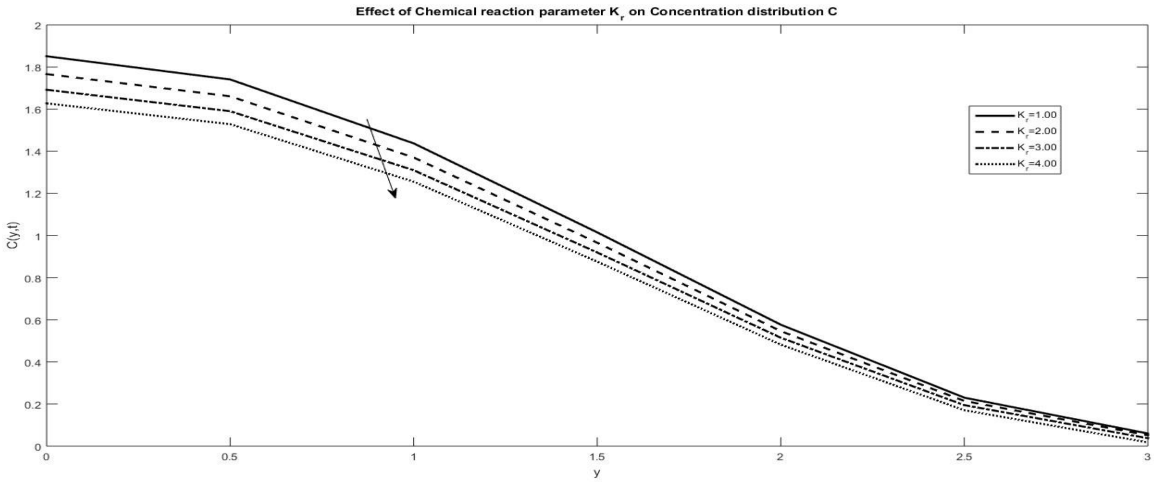

Fig. 6. Effect of Schmidt number S_c on Concentration distribution C

Fig. 7. depict the effect of Grashof number due to mass transfer (G_c) on velocity field. It is observed that the velocity field increases significantly. To this effect, at higher Grashof numbers, the flow at the boundary is turbulent, while at lower Grashof numbers, the flow at the boundary is laminar.

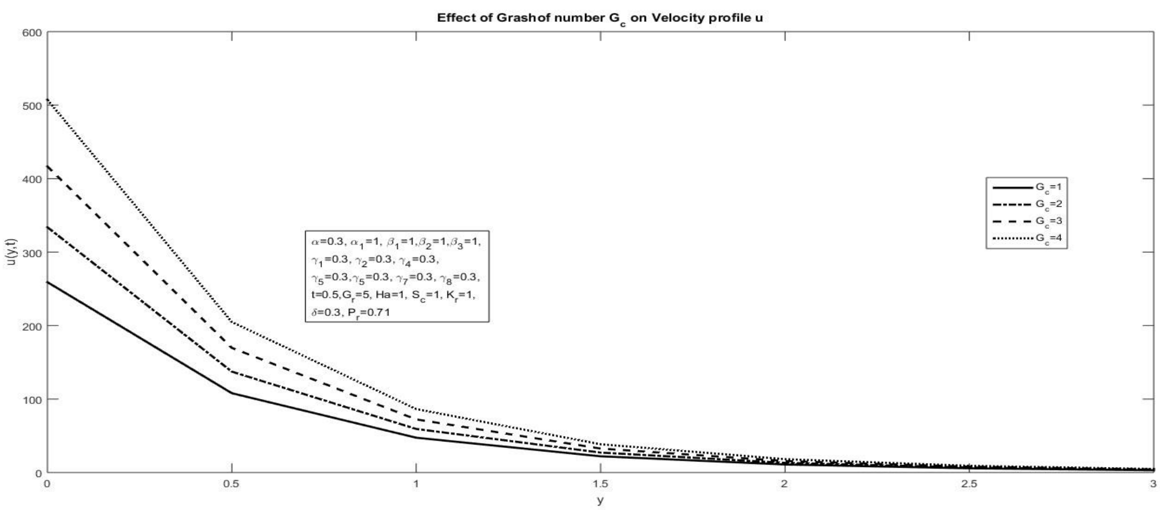

Fig. **7**. Effect of Grashof number G_c due to mass transfer on velocity profile u

Since, the flow is driven by an externally imposed pressure gradient without motion of either plate, Negative and positive pressure gradients increase and decrease the flow rate, respectively. This is illustrated in **Fig. 8.** It is seen that increase in pressure gradient λ enhances the flow significantly.

Pressure Gradient is a physical quantity that describes which direction and at what rate the pressure changes the most rapidly around a particular location.

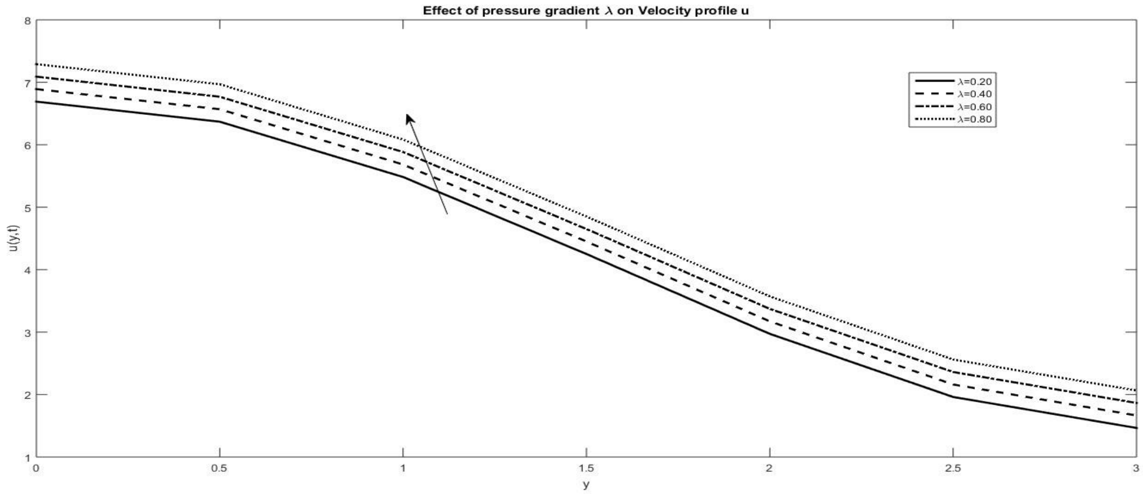

Fig. 8. Effect of pressure gradient parameter λ on Velocity profile u

5. Conclusion

The mass transfer and chemical reaction on unsteady MHD flow of non-Newtonian with pressure gradient has been investigated. The solution for the nonlinear partial differential equations were obtained by the HeLaplace scheme. The effects of flow parameters on velocity and concentration profiles were depicted in figures and discussed. From the results obtained, the following conclusions are made:

(i) For upsurging data of chemical reaction, velocity and concentration fields diminish. (ii) Increasing Hartmann number tend to diminish the velocity profile.

(iii) Strong values of Schmidt number decrease the magnitude of velocity and concentration distribution.

(iv) Increase in Grashof number due to mass transfer accelerate the velocity field. (v) Velocity increases with increase in pressure gradient.

The results of this work are applicable in many industrial producing processes such in the drilling of oil and gas wells, polymer extrusion from dye, glass fibre, paper production and draining of plastics films etc.

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